

Non-equilibrium physics WS 20/21 – Exercise Sheet 11:

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1 Discussion:

- i) What are the physical differences between stationary turbulent solutions and equilibrium solutions of kinetic equations? What are direct and inverse cascades?

2 In-class problems:

2.1 Kolmogorov-Zakharov Spectra

Consider the three wave kinetic equation $\partial_t n(k, t) = I[n](k, t)$ for a statistically homogenous and isotropic system of waves, with power law dispersion $\omega(k) = \omega_0(k/k_0)^z$ with the collision integral given by

$$I[n](k, t) = \frac{1}{2} \int d^d k_1 \int d^d k_2 \left\{ \begin{aligned} & \tilde{w}(k \rightarrow k_1 k_2) \left[n(k_1, t) n(k_2, t) - n(k, t) (n(k_1, t) + n(k_2, t)) \right] \\ & - \tilde{w}(k_1 \rightarrow k_2 k) \left[n(k_2, t) n(k, t) - n(k_1, t) (n(k_2, t) + n(k, t)) \right] \\ & - \tilde{w}(k_2 \rightarrow k k_1) \left[n(k, t) n(k_1, t) - n(k_2, t) (n(k, t) + n(k_1, t)) \right] \end{aligned} \right\} \quad (1)$$

where $\tilde{w}(k \rightarrow k_1 k_2) = (2\pi) \delta(\omega(k) - \omega(k_1) - \omega(k_2)) \delta^{(d)}(k - k_1 - k_2) |V(k, k_1, k_2)|^2$ for a scale invariant matrix element $V(\lambda k, \lambda k_1, \lambda k_2) = \lambda^m V(k, k_1, k_2)$ with the usual symmetry properties $V(k, k_1, k_2) = V(k, k_2, k_1)$.

- i) Show that for a spectrum of the Kolmogorov-Zakharov (KZ) form $n(k, t) = n_{\text{KZ}}(k) = n_0 \left(\frac{k}{k_0}\right)^{-s_0}$ the energy flux $J_k(\Lambda, t) = - \int_0^\Lambda d^d k \omega(k) I[n](k, t)$ through a momentum shell can be expressed as

$$J_k(\Lambda) = -\omega_0 k_0^{d-(\xi+z)} \Omega^{(d)} \int_0^\Lambda dk k^{\xi+z-1} I[n_{\text{KZ}}](k_0),$$

with $\xi = 2d - z + 2m - 2s_0$.

- ii) Determine the scaling exponent s_0 of the KZ spectrum based on the condition that the energy flux $J_k(\Lambda, t) = - \int_0^\Lambda d^d k \omega(k) I[n](k, t)$ through a momentum shell becomes scale (Λ) invariant. Explain why this condition is necessary for $n(k, t) = n_{\text{KZ}}(k)$ to be a stationary turbulent solution.

3 Homework problems:

3.1 Self-similar solutions

Consider the decaying turbulence of a statistically homogenous and isotropic system of waves described by the three wave kinetic equation $\partial_t n(k, t) = I[n](k, t)$.

- iii) Determine the dynamical scaling exponents α, β of the self-similar solutions

$$n(k, t) = (t/t_0)^\alpha n_S \left(k(t/t_0)^\beta \right),$$

of the kinetic equation when a) no energy is injected or removed from the system and b) energy is injected into the system at a constant rate $\dot{\epsilon} = \dot{\epsilon}_0$.

3.2 Stationary turbulence of capillary waves on shallow water

Consider again the three wave kinetic equation $\partial_t n(k, t) = I[n](k, t)$ for a statistically homogenous and isotropic system of waves.

- i) Show that for a spectrum of the Kolmogorov-Zakharov (KZ) form $n(k, t) = n_{\text{KZ}}(k) = n_0 \left(\frac{k}{k_0} \right)^{-s_0}$ the collision kernel $I[n](k, t)$ in Eq. (1) evaluated at $k = k_0$ can be expressed as

$$I[n_{\text{KZ}}](k_0) = \frac{n_0^2}{2\Omega(d)} \int d\Omega_0 \int d^d k_1 \int d^d k_2 \tilde{w}(k_0 \rightarrow k_1 k_2) \left[\left| \frac{k_0}{k_1} \right|^{s_0} \left| \frac{k_0}{k_2} \right|^{s_0} - \left(\left| \frac{k_0}{k_1} \right|^{s_0} + \left| \frac{k_0}{k_2} \right|^{s_0} \right) \right] \times \left[1 - \left| \frac{k_0}{k_1} \right|^\xi - \left| \frac{k_0}{k_2} \right|^\xi \right]$$

with $\xi = 2d - z + 2m - 2s_0$. (Hint: Zakharov transformation)

- ii) Show that in the limit $s_0 \rightarrow m + d$ the energy flux $J_k(\Lambda, t) = - \int_0^\Lambda d^d k \omega(k) I[n](k, t)$ through a momentum shell is given by

$$J_k(\Lambda) = \frac{\omega_0 n_0^2 k_0^d}{2} \int d\Omega_0 \int d^d k_1 \int d^d k_2 \tilde{w}(k_0 \rightarrow k_1 k_2) \times \left[\left| \frac{k_0}{k_1} \right|^{s_0} \left| \frac{k_0}{k_2} \right|^{s_0} - \left(\left| \frac{k_0}{k_1} \right|^{s_0} + \left| \frac{k_0}{k_2} \right|^{s_0} \right) \right] \left[\left| \frac{k_0}{k_1} \right|^{-z} \log \left| \frac{k_0}{k_1} \right| + \left| \frac{k_0}{k_2} \right|^{-z} \log \left| \frac{k_0}{k_2} \right| \right] \quad (2)$$

Now consider as a particular example the dynamics of surface waves in shallow water, as described by a three wave kinetic equation in $d = 2$ dimensions. In this particular case, the dispersion relation is given by $\omega(k) = \omega_0 k^2 / k_0^2$ and the interaction matrix element takes a particularly simple form $V(k, k_1, k_2) = V_0 k^2 / k_0^2$, with $\omega_0 / k_0^2 = \left(\frac{\sigma h}{\rho} \right)^{1/2}$ and $V_0 / k_0^2 = \frac{1}{8\pi} \left(\frac{\sigma}{4\rho h} \right)^{1/4}$ where ρ and σ denote the density and surface tension of the fluid, and h is the height of the container.

- iii) Determine the scaling exponent s_0 of the KZ spectrum for stationary turbulence of capillary waves on shallow water (Check: $s_0 = 4$)
- iv) Show that for two-dimensional dynamics of capillary waves on shallow water, the phase-space integrations in Eq. (2) can be performed according to

$$\int d\Omega_0 \int d^2 k_1 \int d^2 k_2 \tilde{w}(k_0 \rightarrow k_1 k_2) f \left(\left(\frac{k_1}{k_0} \right), \left(\frac{k_2}{k_0} \right) \right) = \frac{(2\pi)^2 k_0^2 V_0^2}{2\omega_0} \int_0^1 dx \frac{f \left(x, \sqrt{1-x^2} \right)}{\sqrt{1-x^2}},$$

- v) Calculate the energy flux through a momentum shell $J_k(\Lambda)$ in Eq. (2) for stationary turbulence of capillary waves on shallow water (Check: $J_k(\Lambda) = 2\pi^3 n_0^2 k_0^4 V_0^2$)
- vi) Show that if energy is injected into the system at a constant rate P , the stationary turbulent solution in the inertial range of the cascade takes the form

$$n_{KZ}(k) = \frac{8P^{1/2}}{\sqrt{\pi}} \left(\frac{\rho h}{\sigma} \right)^{1/4} k^{-4}, \quad (3)$$

3.3 Formulae:

$$x^{a+\epsilon} \simeq x^a + \epsilon x^a \log(x) + \mathcal{O}(\epsilon^2),$$

$$\int_0^1 dx \frac{1}{\sqrt{1-x^2}} \left[x^{-4} \sqrt{(1-x^2)^{-4}} - \left(x^{-4} + \sqrt{1-x^2}^{-4} \right) \right] \left[x^2 \log\left(\frac{1}{x}\right) + (1-x^2) \log\left(\frac{1}{\sqrt{1-x^2}}\right) \right] = 2\pi$$