Navier Stokes equation & fluid dynamics

Discussed dynamics of Newtonian fluids
→ described by conservation equations
for ρ, $\rho\vec{v}$, $\varepsilon + \frac{1}{2}\rho\vec{v}^2$

mass density, momentum density, energy density

(*)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \vec{J}_\rho = 0$$

$$\frac{\partial(\rho v^\alpha)}{\partial t} + \frac{\partial}{\partial x^\beta} J_{\vec{p}}^{\alpha\beta} = 0$$

$$\frac{\partial}{\partial t}\left(\varepsilon + \frac{1}{2}\rho\vec{v}^2\right) + \vec{\nabla}\vec{J}_e = 0$$

closed by constitutive equations
for fluxes J_ρ, $J_{\vec{p}}$, J_e

(□)

$$J_\rho^\alpha = \rho v^\alpha$$ (no non-equilibrium corrections)

$$J_{\vec{p}}^{\alpha\beta} = \Pi^{\alpha\beta} + \rho v^\alpha v^\beta$$

$$J_e^\alpha = \left(\varepsilon + \frac{1}{2}\rho\vec{v}^2\right) v^\alpha + \Pi^{\alpha\beta} v^\beta + J_u^\alpha$$

where

internal energy flux: $J_u^\alpha = -\varkappa \vec{\nabla} T$ $\varkappa \triangleq$ heat conductivity

stress tensor $\Pi^{\alpha\beta} = \left(P - \zeta(\vec{\nabla}\vec{v})\right)\delta^{\alpha\beta} - 2\eta\,\sigma^{\alpha\beta}$

shear stress tensor: $\sigma^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial v^\alpha}{\partial x^\beta} + \frac{\partial v^\beta}{\partial x^\alpha}\right) - \frac{1}{3}\delta^{\alpha\beta}(\vec{\nabla}\vec{v})$

Evaluating the EOMs (A) with the fluxes (B) and assuming $\eta, \zeta \simeq$ const we found

Navier-Stokes equation:

$$\underbrace{\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right]}_{\rho D\vec{v}\ \text{(convective derivative)}} = \underbrace{-\vec{\nabla}P}_{\text{conservative force}} + \overbrace{\zeta\,\vec{\nabla}(\vec{\nabla}\vec{v}) + \eta\left(\Delta\vec{v} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\vec{v})\right)}^{\text{friction forces}}$$

in particular for an incompressible fluid (ρ=const)

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0 \quad \Rightarrow \quad \boxed{\vec{\nabla}\vec{v} = 0}$$

and the Navier Stokes equation simplifies

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\vec{\nabla}P + \eta\,\Delta\vec{v}$$

Q: How can ideal hydrodynamics be obtained from Navier-Stokes equation?

//

Now to investigate what this theory predicts let us first look at the simplest possible problem

<u>Hydrostatics:</u> Stationary (time independent) solutions to the hydrodynamic equations for system at rest

$$\Rightarrow \frac{\partial}{\partial t} x_i = 0 \qquad \vec{v} = 0$$

so in the absence of external forces

$$0 = -\vec{\nabla} P \Rightarrow P = \text{const}$$

which is intuitive as otherwise the fluid would be set into motion

//

Now the problem becomes more interesting if we consider external forces $\vec{f}_{ext}$, whose density need to be added to RHS of NS equation

$$0 = -\vec{\nabla} P + \vec{f}_{ext}$$

$$\Rightarrow \quad \vec{\nabla} P = \vec{f}_{ext}$$

So for instance if we consider gravity: $\vec{f}_{ext} = -g \rho \vec{e}_z$

this becomes a differential equation

$$\frac{\partial}{\partial z} P\big(\rho(z), e(z)\big) = -g \rho(z)$$

Now to solve this ODE we further need an EOS so e.g. for an ideal gas

$$pV = N k_B T$$

$$p = \frac{N}{V} k_B T = \rho \frac{k_B T}{m}$$

where $T = T(\rho, e)$. So assuming $T \simeq$ const we get

$$\frac{k_B T}{m} \frac{\partial}{\partial z} \rho = -g \rho$$

$$\Rightarrow \quad \frac{\partial}{\partial z} \rho = -\frac{mg}{k_B T} \rho$$

Solution is given by

$$\rho(z) = \rho_0 \, e^{-\frac{mg}{k_B T} z}$$

//

Now let us have a closer look at dynamical systems and first investigate the effects of dissipative forces in an incompressible fluid

Naive expectation is that frictions will slow down the fluid and convert fluid kinetic energy to internal energy

$$E_{kin}(t) = \int d^3r \; \frac{1}{2} \rho(t,\vec{r}) \, \vec{v}^{\,2}(t,\vec{r}) \qquad E_{int}(t) = \int d^3\vec{r} \; \varepsilon(t,\vec{r})$$

Since

$$\frac{\partial}{\partial t}\left(\varepsilon + \frac{1}{2}\rho \vec{v}^{\,2}\right) + \vec{\nabla} \vec{J}_E = 0$$

we have $\int d^3\vec{r} \; \vec{\nabla} \vec{J}_E = 0$ (boundary terms vanish)

$$\boxed{\frac{d}{dt} E_{kin}(t) = -\frac{d}{dt} E_{int}(t)}$$

Now lets look at the change of $E_{kin}(t)$ for an incompressible fluid

$$\frac{\partial}{\partial t} E_{kin}(t) = \frac{\partial}{\partial t} \int d^3\vec{r} \, \frac{1}{2} \rho(t,\vec{r}) \, \vec{v}^2(t,\vec{r})$$

$$= \rho \int d^3\vec{r} \quad v^\alpha(t,\vec{r}) \, \frac{\partial v^\alpha(t,\vec{r})}{\partial t}$$

from NS equation we know $\rho \frac{\partial v^\alpha}{\partial t}$

$$= \int d^3\vec{r} \, v^\alpha \left[-\rho \left(v^\beta \frac{\partial}{\partial x^\beta} \right) v^\alpha - \frac{\partial}{\partial x^\alpha} P + \eta \frac{\partial^2}{\partial x^\beta \partial x^\beta} v^\alpha \right]$$

now inspect each term separately

$$\rho v^\alpha \left(v^\beta \frac{\partial}{\partial x^\beta} \right) v^\alpha = v^\beta \frac{\partial}{\partial x^\beta} \left(\frac{1}{2} \rho \vec{v}^2 \right)$$

$$= \frac{\partial}{\partial x^\beta} \vec{v}^\beta \left(\frac{1}{2} \rho \vec{v}^2 \right) - \frac{1}{2} \rho \vec{v}^2 \underbrace{\left(\frac{\partial}{\partial x^\beta} v^\beta \right)}_{=0}$$

So the result is given by a total derivative, which vanishes under integration as boundary terms can be neglected if $\vec{v}, \rho \to 0$ as $|\vec{r}| \to \infty$

Similarly

$$v^\alpha \frac{\partial}{\partial x^\alpha} P = \frac{\partial}{\partial x^\alpha} v^\alpha P - P \underbrace{\left(\frac{\partial}{\partial x^\alpha} v^\alpha\right)}_{=0}$$

also vanishes under the integral, such that for an ideal incompressible fluid there is no change of the total fluid kinetic energy

Now for viscous fluids we also have to consider

$$\eta\, v^\alpha \frac{\partial}{\partial x^\beta \partial x^\beta} v^\alpha = \eta \frac{\partial}{\partial x^\beta}\left(v^\alpha \frac{\partial v^\alpha}{\partial x^\beta}\right)$$

$$- \eta \left(\frac{\partial v^\alpha}{\partial x^\beta}\right)\left(\frac{\partial v^\alpha}{\partial x^\beta}\right)$$

where for $\eta = \text{const}$ as in NS eqn the first term again represents a total derivative that vanishes under integration

$$\Rightarrow \quad \frac{\partial}{\partial t} E_{kin}(t) = -\eta \int d^3r \left(\frac{\partial v^\alpha}{\partial x^\beta}\right)^2$$
$$\leq 0$$

In viscous fluids kinetic energy E_{kin} is dissipated into internal energy E_{int} as a consequence of the viscous friction forces

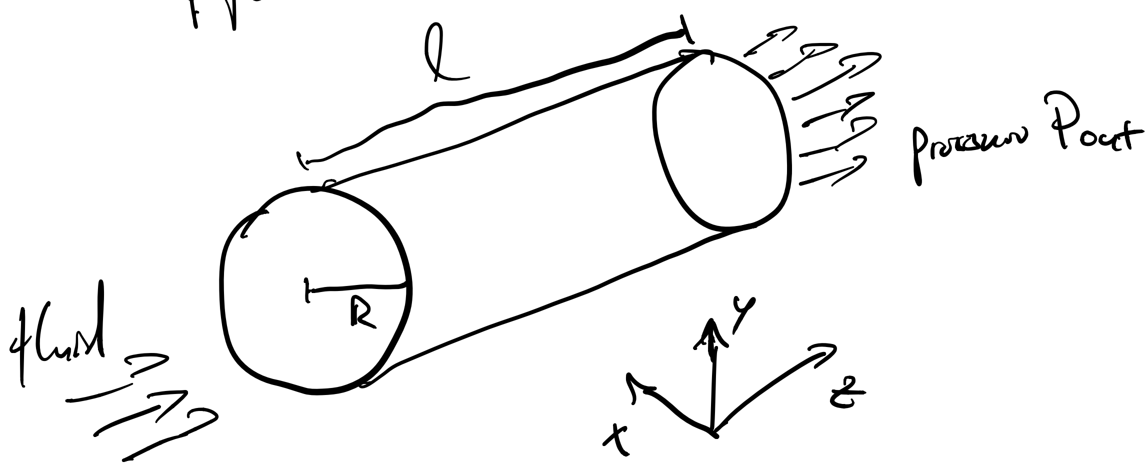
//

Generally fluid dynamics is rather complicated but powerful to describe complex phenomena such as
weather, aerodynamics, turbulence, ...
with detailed predictions relying on sophisticated numerical simulations

Some classic problems of fluid dynamics can be solved analytically

Stationary flow in a pipe:

We consider an incompressible fluid flowing in a cylindrical pipe



such that fluid is moving in the z-direction

$$\vec{V} = V_z(t, x, y, z)\,\vec{e}_z$$

So far an incompressible fluid

$\rho = \text{const}$

$$\underbrace{\frac{\partial}{\partial t}\rho}_{=0 \text{ (stationary)}} + \vec{\nabla}(\rho\vec{v}) = 0 \Rightarrow \vec{\nabla}\vec{v} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial z} V_z(t,x,y,z) = 0$$

Since t dependence also drops out as we are interested in stationary solution

$$V_z(t,x,y,z) = V_z(x,y).$$

By rotational symmetry in the x,y plane, we can expect

$$V_z(x,y) = V_z(r) \qquad r = \sqrt{x^2+y^2}$$

Now to determine $V_z(r)$, we look at NS equation (z-component)

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\vec{\nabla}P + \zeta\,\vec{\nabla}(\vec{\nabla}\vec{v}) + \eta\left(\Delta\vec{v} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\vec{v})\right)$$

so by using $\frac{\partial \vec{v}}{\partial t} = 0$, and incompressibility

$$0 = -\frac{\partial}{\partial z}P + \eta\,\Delta V_z$$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial z}P = \eta\,\Delta V_z}$$

Now if we assume $\eta \approx \text{const}$, we can take a further derivative wrt z

$$\frac{\partial^2}{\partial z^2}P = \eta\,\frac{\partial}{\partial z}\Delta V_z = \eta\,\Delta\underbrace{\left(\frac{\partial}{\partial z}V_z\right)}_{=0} = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial z}P = \text{const} = \frac{P_{out} - P_{in}}{\ell} = -\frac{\Delta P}{\ell}}$$

So to determine the velocity profile we need to solve

$$\eta \Delta V_z(r) = -\frac{\Delta P}{\ell}$$

Expressing Δ in cylindrical coordinates

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)V_z(r) = -\frac{\Delta P}{\eta \ell}$$

which can now be solved by integration

$$r\frac{\partial}{\partial r}V_z(r) = -\frac{1}{2}\frac{\Delta P}{\eta \ell}r^2 + a$$

a integration constant

general solution

$$\boxed{V_z(r) = -\frac{1}{4}\frac{\Delta P}{\eta \ell}r^2 + a\log(r) + b}$$

Still need to fix integration constants by implementing boundary conditions

1) $a \log(r) \xrightarrow{r \to 0} \infty$ unless $a = 0$

so we need $a = 0$ for the fluid velocity to be finite in the center of the pipe

2) If we consider the outer boundary

fluid will encounter friction with the wall unless $\vec{v} = 0$ as wall is approached

Stationary flow has $v_z(r) \to 0$ as $r \to R$ where the walls of the cylinder are located

$$V_z(R) = b - \frac{1}{4}\frac{\Delta P}{\eta l} R^2 \stackrel{!}{=} 0$$

Solution to flow in a pipe

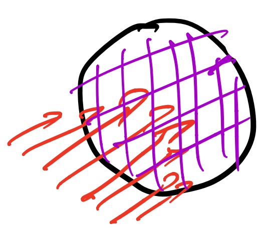
„Poiseuille flow"

$$\boxed{V_z(r) = \frac{1}{4}\frac{\Delta P}{\eta l}\left(R^2 - r^2\right)}$$

Now what is interesting about this is to calculate how much fluid is transported through the pipe

<u>Discharge:</u>

$$Q = \underbrace{\int_{\circ} d^2 r_s \; \vec{e}_s}_{\text{cross-sectional area of the pipe}} \cdot \underbrace{(\rho \vec{v})}_{\text{mass flux}}$$



$$Q = 2\pi\rho \int dr\, r\, v_z(r) = \frac{\pi \Delta P}{8\eta l} R^4 \rho$$

$\Rightarrow$ Bigger pipes transport more fluid $Q \propto R^4$

1) because they are bigger $\propto R^2$

2) because the slow flow near the wall has a smaller impact on the fluid transport in the center