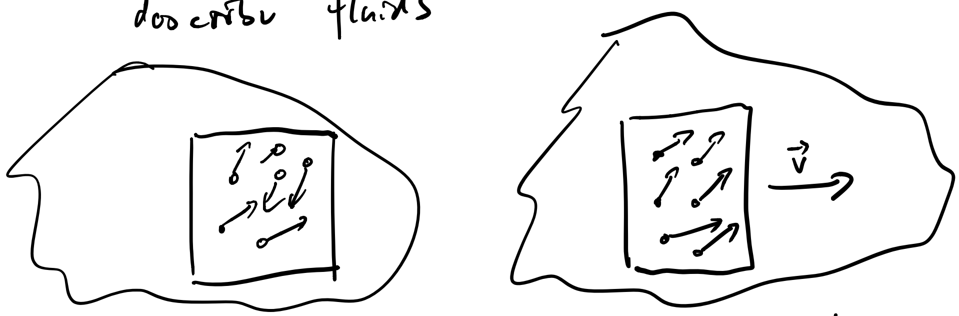
1.4.3 Linear Markovian Dynamics of simple fluids

So far we discussed linear transport phenomena in systems at rest (e.g. diffusion, conduction thermoelectric effects)

Need to consider generalization to systems which can move as a whole, to describe fluids



sub-System at rest

$\vec{P} = 0$

sub-System not at

$\vec{P} \neq 0$

Non-zero value of conserved quantity ($\vec{P} \neq 0$) needs to be accounted for and included into set of macroscopic variables

$\rightarrow$ balance equation for momentum density + constitutive relations for momentum flux

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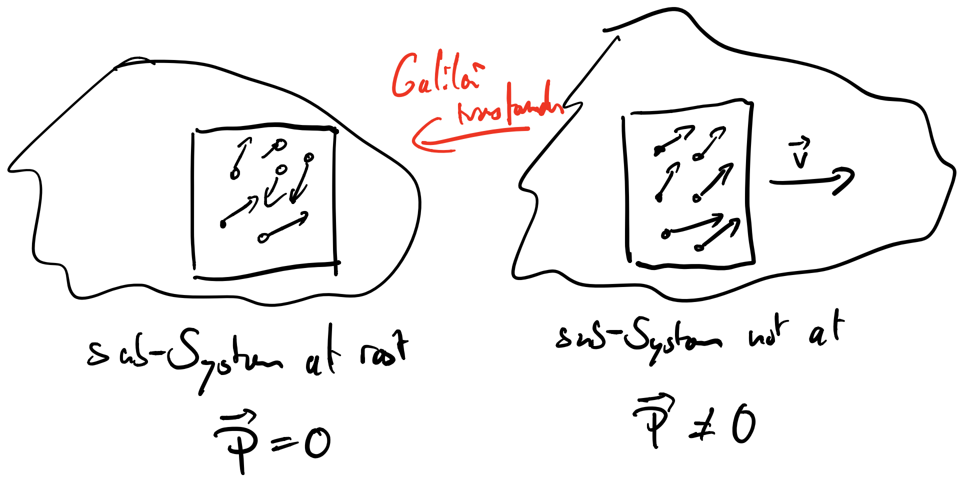
Exploit the fact that we can relate moving and resting system by __Galileo transformation__



$\Rightarrow$ always possible to define a comoving frame „__local rest frame__" where sub-system is at rest $(\vec{P} = 0)$

Denote frames as

$\vec{R}_V$ local rest frame (LRF)

$\vec{R}_0$ laboratory frame

Since the physics is the same irrespective of the frame that we use to describe it consider first the situation in LRF and infer what follows in laboratory frame

	LRF	LAB
<u>Energy:</u>	$E = U$	$E = U + \frac{1}{2}\frac{\vec{P}^2}{2M}$
	describes statistical motion of particles	includes COM motion

Since $\vec{P}$ appears explicitly, we need to include it in macroscopic description. Since the internal properties of the fluid are the same, the entropy describing the thermodynamic properties is frame independent

$$\Rightarrow S(E, N, V, \vec{P}) = S_{TD}(\underbrace{E - \frac{1}{2}\frac{\vec{P}^2}{2M}}_{=U}, N, V)$$

for discrete system

Similarly for continuous systems we need to keep track of the momentum density $\vec{P}(t,\vec{r})$

but usually in non-relativistic Hydrodynamics use $\vec{v}$ (velocity) rather than momentum

$$\vec{P}(t,\vec{r}) = \rho(t,\vec{r})\,\vec{v}(t,\vec{r})$$

$$\mathcal{E}_{com}(t,\vec{r}) = \frac{1}{2}\rho(t,\vec{r})\,\vec{v}^{\,2}(t,\vec{r})$$

where for a single component fluid

$$\rho(t,\vec{r}) = m\, n(t,\vec{r})$$

So if we consider the energy density, we can always distinguish between total, internal and fluid kinetic energy density

LRF

$$\mathcal{E}_{tot}(t,\vec{r}) = \underbrace{\mathcal{E}(t,\vec{r})}_{\text{internal}}$$

LAB

$$\mathcal{E}_{tot}(t,\vec{r}) = \underbrace{\mathcal{E}(t,\vec{r})}_{\text{internal}} + \underbrace{\frac{1}{2}\rho(t,\vec{r})\,\vec{v}^{\,2}(t,\vec{r})}_{\text{fluid kinetic}}$$

Note that for a simple fluid energy, momentum and particle number conservation imply that $\mathcal{E}_{tot}$, $\vec{P}$, n are densities of conserved quantities

$\rightarrow$ can anticipate balance equations for $\mathcal{E}_{tot}$, $\vec{P}$, n

Now to construct linear constitutive relations we need equilibrium fluxes $\mathcal{J}_i^{eq}$ and affinities $\vec{F}_i = \vec{\nabla} Y_i$ for $\mathcal{E}_{tot}$, $\vec{P}$, n

Since in the presence of $\vec{P} \neq 0$ entropy functional is modified

$$S(\mathcal{E}_{tot}, n, \vec{P}) = S_0(\underbrace{\mathcal{E}_{tot} - \frac{\vec{P}^2}{2mn}}_{\equiv e}, n)$$

first determine conjugate intensive quantities

$$ds = \underbrace{\left.\frac{\partial s_0}{\partial \varepsilon}\right|_n}_{=\frac{1}{T}} \left(d\varepsilon_{tot} - \frac{1}{2m} d\left(\frac{\vec{p}^2}{n}\right)\right) + \underbrace{\left.\frac{\partial s_0}{\partial n}\right|_\varepsilon}_{=-\frac{\mu}{T}} dn$$

$$= \frac{1}{T} d\varepsilon_{tot} - \frac{1}{T}\frac{\vec{p}}{mn} d\vec{p} + \frac{1}{T}\frac{\vec{p}^2}{2mn^2} dn - \frac{\mu}{T} dn$$

$$= \underbrace{\frac{1}{T}}_{\equiv Y_E} d\varepsilon_{tot} - \underbrace{\frac{\vec{v}}{T}}_{\equiv Y_{\vec{p}}} d\vec{p} - \underbrace{\left(\frac{\mu}{T} - \frac{\frac{1}{2}m\vec{v}^2}{T}\right)}_{\equiv Y_n = -\frac{\mu_{\vec{p}}}{T}} dn$$

Now since $\vec{p} = mn(t,\vec{r})\,\vec{v}(t,\vec{r})$

a change in momentum can be achieved by adding particles or changing the velocity.

Hence it is also useful to re-express in the terms of ε_{tot}, $\vec{v}$ and n

$$d\vec{p} = m\,dn\,\vec{v} + mn\,d\vec{v}$$

$$ds = \frac{1}{T}\,dE_{tot} - \frac{\vec{p}}{T}\,d\vec{v} - \underbrace{\left(\frac{\mu}{T} + \frac{1}{2}\frac{m\vec{v}^2}{T}\right)}_{\frac{\mu_{\vec{v}}}{T}} dn$$

to consider the effects of a frame transformation

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Now we can see for instance that if we look at the Pressure

Gibbs-Duhem relation:

$$P = Ts - \varepsilon + \mu n$$

$$= Ts - \varepsilon - \frac{1}{2}\rho\vec{v}^2 + \frac{1}{2}\underbrace{\rho}_{mn}\vec{v}^2 + \mu n$$

$$= Ts - \varepsilon_{tot} + \mu_{\vec{v}}\, n$$

indicating that pressure is the same in any reference frame

"Now we have the intensive quantities $y_i(t,\vec{r})$ for $\varepsilon, n, \vec{p}$ from which we can construct the affinities

$$F_i^{\alpha}(t,\vec{r}) = \frac{\partial}{\partial x^{\alpha}} y_i(t,\vec{r}) \quad i = \varepsilon, n$$

$$F_i^{\alpha\beta}(t,\vec{r}) = \frac{\partial}{\partial x^{\alpha}} y_i^{\beta}(t,\vec{r}) \quad i = \vec{p}$$

However to obtain a closed description, we also need equilibrium fluxes

$$J_i^{\circ\alpha}$$

and equations of state

Now to construct equilibrium fluxes we will first consider LRF and then derive general form in LAB frame by considering transformation properties

Equilibrium fluxes (LRF)

$\vec{J}_E, \vec{J}_N$ vectorial fluxes

$\underline{\underline{J}}_{\vec{p}} = J^{\alpha\beta}_{\vec{p}}$ tensorial flux of rank 2

$J^{\alpha\beta}_{\vec{p}}$ describes flux of p^α in x^β direction

Exploiting rotational invariance

of equilibrium system in LRF

$\Rightarrow$ no preferred direction

$$\vec{\mathfrak{Z}}_E\Big|_{LRF} = \vec{\mathfrak{Z}}_N\Big|_{LRF} = 0$$

$$\mathfrak{Z}^{\alpha\beta}_{\vec{p}} = \delta^{\alpha\beta} C_{\vec{p}}$$

where $C_{\vec{p}}$ describes an

isotropic flux of momenta

in all directions

Q: What is the constant $C_{\vec{p}}$?

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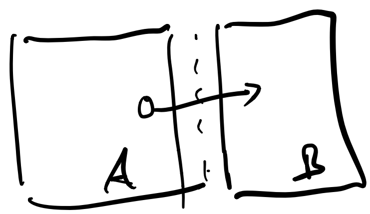
Now to answer this lets consider a fluid in LTE and sub-divide into two cells



Surface S

momentum flux $J_{\vec{p}}^{\alpha\beta}$ correspond to particles crossing the boundary

change of momentum
$$\frac{d\vec{P}_A^{\alpha}}{dt} = \int_S d^2 r_s \; \vec{e}_s^{\,\beta} \, J_{\vec{p}}^{\alpha\beta}$$

$$\overset{eq}{=} S \, C_{\vec{p}} \, \vec{e}_s^{\,\alpha}$$

$$= \vec{F}_{B\to A}^{\alpha}$$

force exerted by B on A

So ~~constant~~ $C_{\vec{p}} = \dfrac{\vec{F}_{B \to A} \cdot \vec{e}_s}{S}$

is the force per unit area which is given by the thermodynamic pressure P

So collecting everything we have the following equilibrium fluxes in the local rest frame

$$\boxed{\vec{J}_E \Big|_{eq} = 0 \,, \quad \vec{J}_N \Big|_{eq} = 0 \quad J^{\alpha\beta}_{\vec{p}} \Big|_{eq} = P \, \delta^{\alpha\beta}}$$

=

Now to describe motion of a fluid we need to know the constitutive relations in a general frame (e.g. lab frame)

$\Rightarrow$ will be obtained by by performing Galilei transformation

We first consider infinitesimal transformation between

$R_{\vec{v}'}$ where fluid cell moves with $\vec{v}-\vec{v}'$

$R_{\vec{v}'+d\vec{v}'}$ where fluid cell moves with $\vec{v}-\vec{v}'-d\vec{v}'$

Start by looking at the densities

$\mathcal{E}_{tot}, n, \vec{p}$

measured in $R_{\vec{v}'}$

$$\mathcal{E}_{tot} = \mathcal{E} + \frac{1}{2}\rho(\vec{v}-\vec{v}')^2$$
$$n = n$$
$$\vec{p} = \rho(\vec{v}-\vec{v}')$$

measured in $R_{\vec{v}'-d\vec{v}'}$

$$\mathcal{E}_{tot} = \mathcal{E} + \frac{1}{2}\rho(\vec{v}-\vec{v}'-d\vec{v}')^2$$
$$n = n$$
$$\vec{p} = \rho(\vec{v}-\vec{v}'-d\vec{v}')$$

So if we consider the differential change denoting $\vec{w} = (\vec{v}-\vec{v}')$ $\quad d\vec{w} = -d\vec{v}'$

$$\boxed{\begin{aligned} &d(\underbrace{\varepsilon + \tfrac{1}{2}\rho \vec{w}^2}_{\varepsilon_{tot}}) = \rho \vec{w}\, d\vec{w} = \underline{\vec{p}}\, d\vec{w} \\ &d\underline{n} = 0 \\ &d\underline{\vec{p}} = \rho\, d\vec{w} = m\, \underline{n}\, d\vec{w} \end{aligned}}$$

Now we also have to transform the fluxes/currents where two effects have to be accounted for

1) Currents that are already present in $R_{\vec{v}}'$ will change in $R_{\vec{v}+d\vec{v}}'$ in the same way that the corresponding densities do

2) Densities at rest in $R_{\vec{v}}'$ will move with $-d\vec{v}'$ in $R_{\vec{v}+d\vec{v}}'$ and contribute additionally to the flux

Need to recall that $\mathfrak{J}_{\vec{p}}^{\alpha\beta}$ denotes momentum p^α flux in x^β direction

particles at rest flow

$$d\mathfrak{J}_N^\alpha = n\, dw^\alpha$$

momentum p^β flux in x^α direction contributes to energy flux

$$d\mathfrak{J}_E^\alpha = \mathfrak{J}_{\vec{p}}^{\beta\alpha}\, dw^\beta + \left(\mathcal{E} + \tfrac{1}{2}\vec{w}^2\right) dw^\alpha$$

energy at rest flows

particles flux in x^α direction contributes to momentum p^β flux

$$d\mathfrak{J}_{\vec{p}}^{\alpha\beta} = m\,\mathfrak{J}_n^\beta\, dw^\alpha + \rho\, w^\alpha\, dw^\beta$$

momentum at rest flows

Differential equations for behavior of densities $\varepsilon_{tot}, n, \vec{p}$ and fluxes $\vec{J}_E, \vec{J}_n, \underline{J}_{\vec{p}}$ under change of frame

$\Rightarrow$ solve starting from constitutive relations in LRF to obtain constitutive relations in any frame