V. Stochastic processes

We will discuss basic properties of stochastic processes, focusing primarily on Brownian motion of classical particles

Before we set up the formalism, we first note that most physical systems are deterministic and in principle there is no randomness involved in describing the dynamics of the system

However to describe the dynamics of complex many body systems, we need to keep track of a large number of degrees of freedom

Naturally disregarding some degrees of freedom introduces an element of randomness in the theoretical description

→ Stochastic processes are frequently used to model dynamics of complex many-body systems

Some examples of physically relevant systems which feature fluctuations due to coarse graining assumptions include e.g.

- thermal fluctuations in fluids
- Diffusion processes
- Markov-Chain Monte-Carlo simulations

Besides their importance in physics, stochastic processes also play an important role in chemistry, biology & quantitative finance

e.g. Black-Scholes model

We will focus on the simplest possible example of one dimensional Brownian motion which serves as a good example to illustrate the most important principles

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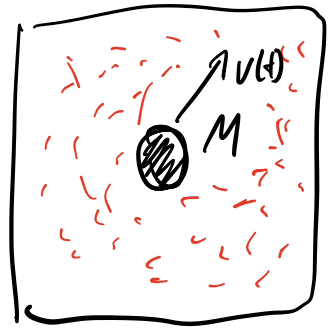
Brownian motion (1D)



We consider a heavy particle with mass M moving through a fluid of light particles

We wish to describe the dynamics of the heavy particle, without explicitly keeping track of the light particles, which we want to eliminate from the theoretical description

Generally for a classical particle, the motion is described by Newtons law

$$M \frac{dv}{dt} = \sum \vec{F}$$

where $\sum \vec{F}$ describes the sum of all forces exerted by the light particles onto the heavy particle. If we don't keep track of the light particles explicitly, there will be randomness in the force $\vec{F}$ experienced by the heavy particle.

Now if the heavy particle is moving with a velocity $v(t)$ in the rest frame of the fluid, on average there should be a drag force slowing down the particle, which we will model as

$$F_{Drag} = - M\gamma\, v(t)$$

where e.g. for a spherical object embedded in a viscous fluid, this is described by Stokes friction

<u>friction coefficient</u>: $\gamma = \frac{6\pi\eta R}{M}$

While this describes the average effect of interactions between light and heavy particles, we know that microscopically the force on the heavy particle is due to collisions with light particles, which accelerate/decelerate the heavy particle.

Each individual scattering event has an outcome that is in general different from the average, and we can not predict the outcome of individual scatterings without further knowledge of the microscopic dynamics of light particles.

Basic idea of Langevin & Brown is to model the fluctuations in individual collisions by a <u>stochastic force</u>

$$F_L(t)$$

which is randomly distributed according to a statistical distribution, but different in each particular realization of the system

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Properties of stochastic force $F_L(t)$

Since the average effect of collisions is already accounted for by the drag force we have

$$\langle F_L(t) \rangle = 0$$

where $\langle . \rangle$ denotes an average over different microscopic realizations of the system

However, the stochastic force describes the fluctuations

auto-correlation function:

$$\langle F_L(t)\, F_L(t') \rangle = \mathcal{K}(t,t')$$

for one-dimensional Brownian motion

Note that for higher dimensional systems or stochastic processes of many variables, the auto-correlation function is a matrix

$$\langle F_L^i(t) F_L^j(t') \rangle = \mathcal{K}_{ij}(t,t')$$

where e.g. for 3-D Brownian motion

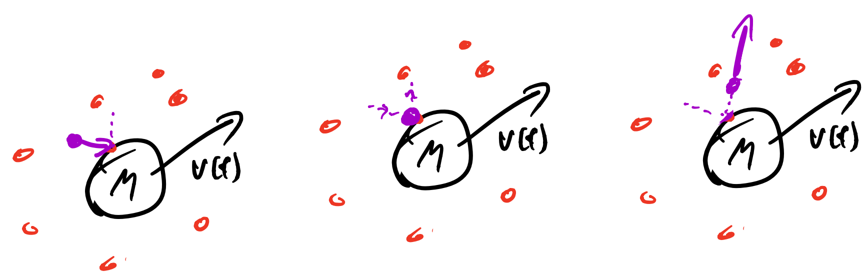
$$\mathcal{K}_{ij}(t,t') = \delta_{ij}\, \mathcal{K}(t-t')$$

Now if we consider the fluid as stationary, e.g. in global equilibrium the statistical properties of the force should be invariant under time translations

$$\rightarrow \quad \mathcal{K}(t,t') = \mathcal{K}(t-t')$$

auto-correlation function only depends on the time difference $t-t'$

Next to determine further properties of the auto-correlation function, we need to consider the underlying microscopic dynamics of the interaction of light and heavy particles, that we intend to describe by the stochastic force



$\longleftarrow \sim \tau_{coll} \longrightarrow$

Generally when the heavy particle interacts with a light particle, there will be a force exerted on the heavy particle over a time scale $\sim \tau_{coll}$ (collision time)

Since on time scales $|t-t'| < \tau_{coll}$ the force F_L is due to the interaction with a single particle, we can expect a

<u>coherent force</u> on the microscopic time scale of individual collisions

$$\mathcal{K}(t-t') > 0 \quad \text{for} \quad |t-t'| \lesssim \tau_{coll}$$

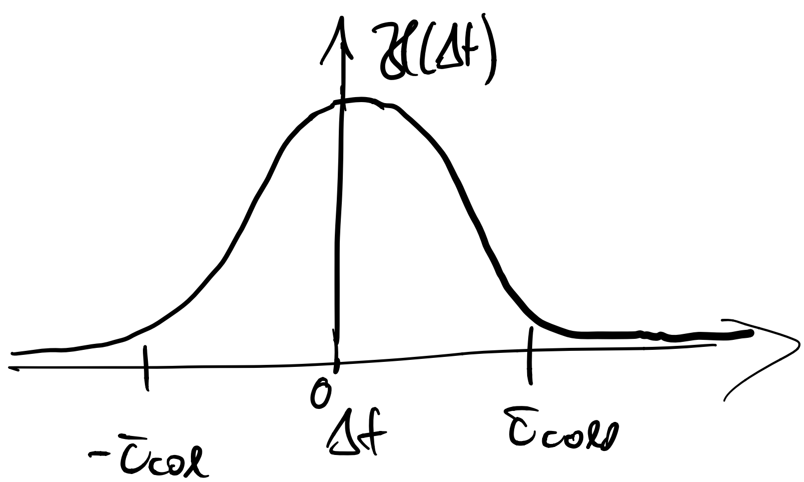
Subsequently for $|t-t'| \gg \tau_{coll}$, the heavy particle will interact with many <u>different</u> light particles

if the outcome of each individual scattering event is independent of each other (e.g. molecular chaos), the heavy particle will experience a different uncorrelated force during each interaction.

Stochastic force is incoherent over time scales $|t-t'| \gg \tau_{coll}$

$$\mathcal{K}(t-t') \to 0 \quad \text{for} \quad |t-t'| \gg \tau_{coll}$$

Based on the limiting behaviour, we can expect the following characteristic behaviour of the auto-correlation function for Brownian motion



where the net-effect of the fluctuations on short time scales $\sim\tau_{coll}$, can be quantified by

$$\int_{-\infty}^{+\infty} d\Delta t \; \mathcal{K}(\Delta t) = 2 D_v M^2$$

which defines velocity diffusion constant D_v

In particular if we are interested in the dynamics on time scales $t \gg \tau_{coll}$ it should be sufficient to approximate

$$\mathcal{K}(\Delta t) = 2 D_v M^2 \delta(\Delta t)$$

in the spirit of a coarse grained description, where the interactions are instantaneous on macroscopic scales

Note further, that generally in order to specify the stochastic process completely, we also need to specify the higher order correlation functions of the stochastic force

$$\langle F_L(t_1) \dots F_L(t_N) \rangle$$

One possibility is to assume Gaussian distribution, whereby all N-point functions are determined by the 1 and 2-point correlation functions, which can be justified to some extent by the central limit theorem.

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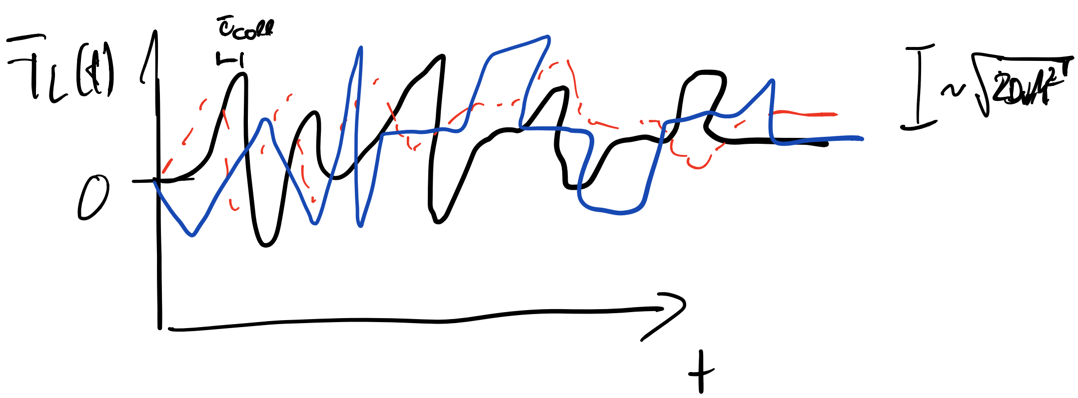
Based on the above considerations, the dynamics of the heavy particle is then described by

$$\boxed{M \frac{dV}{dt} = -M\gamma V(t) + F_L(t)}$$

which involves the stochastic force $F_L(t)$ that will be different in every single microscopic realization of the system



Due to appearance of $F_L(t)$ the EOM classifies as

→ <u>stochastic differential equation</u> (SDE)

While in every single microscopic realization the stochastic force $F_L(t)$ takes on a particular value at each time and the dynamics is described by an ordinary differential equation (ODE) for this particular force, the macroscopic properties of the system are described by averaging over all possible microscopic realizations.

So in order to solve SDE, we have two possibilities

1) Solve ODE for general force $F_L(t)$ and subsequently average over all possible realizations

2) Directly construct EOMs for expectation values and solve associated ODEs

Generally this is a difficult task, however in this particular case, the SDE is of a particularly simple form as it is <u>linear in $v(t)$</u> and the <u>stochastic force</u> $F_L(t)$ is <u>independent of the velocity $v(t)$</u> which allows us to solve the SDE.

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Evolution of the velocity

Starting from SDE

$$\boxed{M \frac{dV}{dt} = -M\gamma V(t) + F_L(t)}$$

we obtain that for each microscopic realization (where $F_L(t)$ assumes a particular form)

$$V(t) = \underbrace{V(t_0)\, e^{-\gamma(t-t_0)}}_{\text{homogeneous solution}} + \underbrace{\frac{1}{M}\int_{t_0}^{t} dt'\, e^{-\gamma(t-t')} F_L(t')}_{\text{inhomogeneous solution}}$$

Such that on average we get

$$\langle V(t)\rangle = \langle V(t_0)\rangle\, e^{-\gamma(t-t_0)}$$

describing an exponential relaxation of the initial velocity towards zero on a time scale $\sim 1/\gamma$, due to Drag force

Similarly, we can obtain the same result by directly looking at evolution equation for the expectation value

$$M \frac{d}{dt} \langle v(t) \rangle = - M \gamma \langle v(t) \rangle$$

yielding the same solution