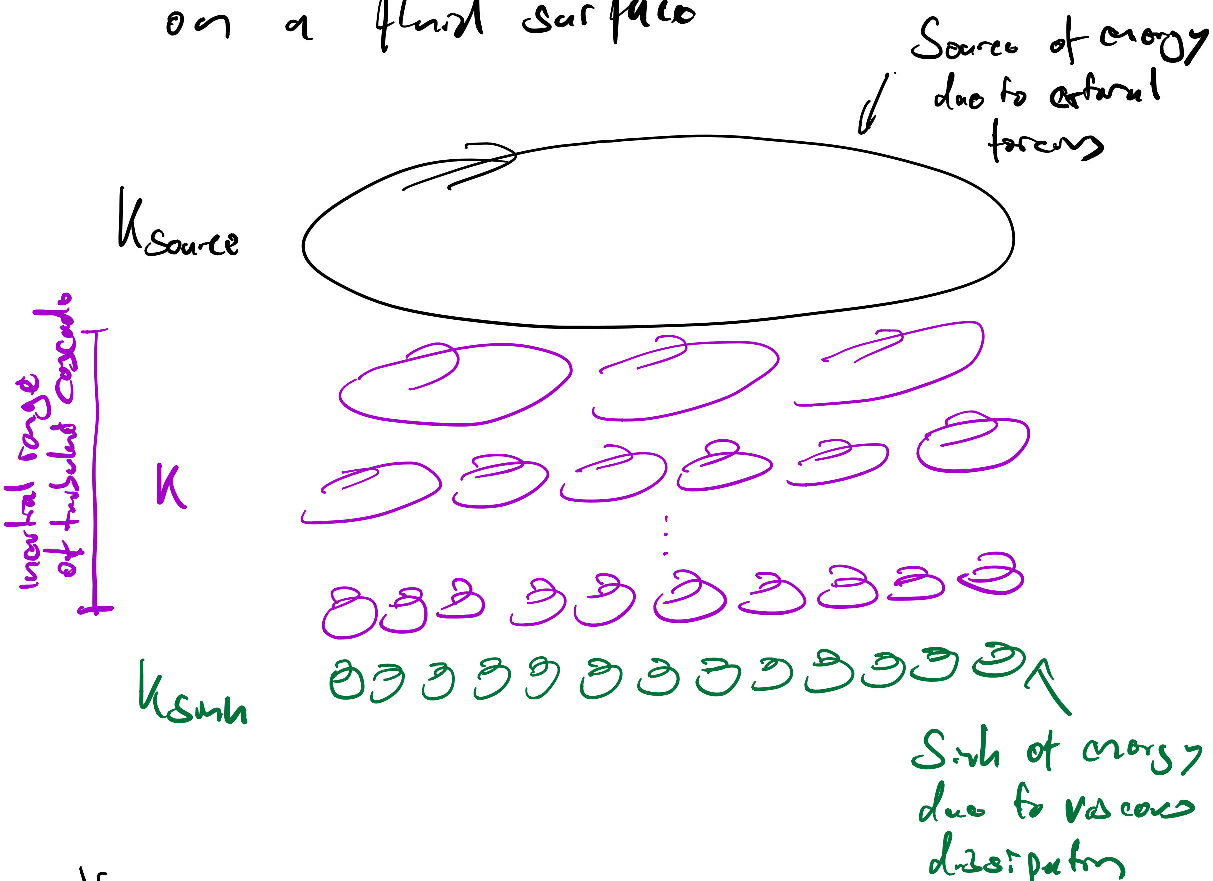
Discussed stationary turbulent solutions of wave-kinetic equations

→ Describe transport of conserved quantity across large separation of scales

Best example is the Richardson cascade of weak wave turbulence of capillary waves on a fluid surface



Dynamics of weakly non-linear waves described by wave-kinetic equation (analogous to BTE for classical particles) for statistically homogeneous and isotropic systems

$$\underbrace{\frac{\partial}{\partial t} n(k,t)}_{\text{number of waves with wave number } k} = \underbrace{I[n](k,t)}_{\text{collision integral for weakly non-linear wave interactions}} + \text{source/sink}$$

Effects of source/sink terms localized in K-space, so in inertial range of turbulent cascade

$$\frac{\partial}{\partial t} n(k,t) = I[n](k,t)$$

Stationary solutions described by zeros of collision integral

$$I[n](k,t) = 0$$

within inertial range of wave numbers k well separated from source and sink

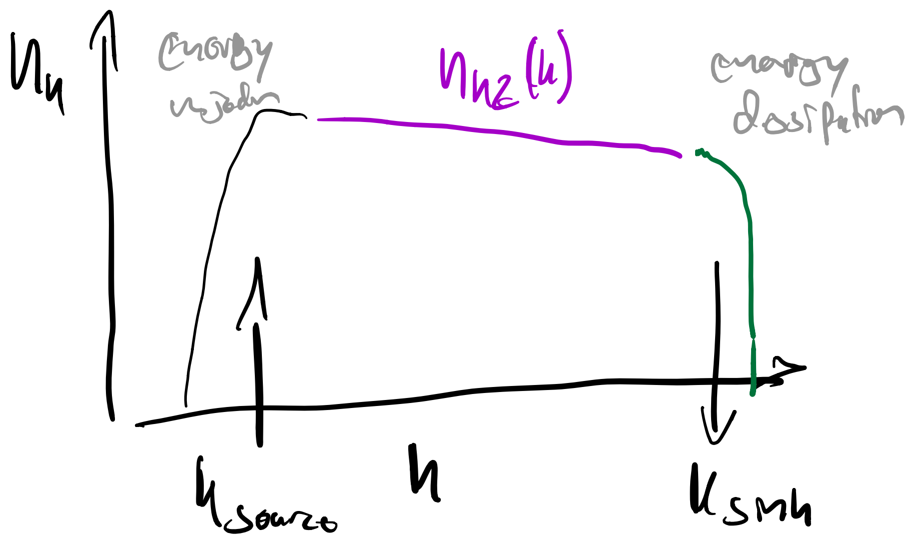
Determined the solutions for scale invariant systems with the help of Zakharov (scale) transformations as Kolmogorov-Zakharov spectra

$$\boxed{n_{KZ}(k) = n_0 \left(\frac{k}{k_0}\right)^{-s_0}}$$

with e.g. $\boxed{s_0 = m+d}$ for 3-wave interactions, which

satisfy balance $I[n_{KZ}](k,t)=0$
in the marked range of wave-numbers

Generally we can therefore
expect that the following
typical spectrum



for a direct cascade transporting
energy from a large scale k_{source}
to a small scale $k_{sink} \gg k_{source}$

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Now to understand how the Kolmogorov-Zakharov (KZ) spectra are related to the energy transfer from k_{source} to k_{sink}, we will consider

Balance equations in k-space & energy transfer along the cascade

We will for simplicity focus on 3-wave interactions, where interactions can change the number of wave excitations, and energy is the only relevant conserved quantity

Globally the energy conservation law reads

$$\frac{dE}{dt} = \dot{E}_{in} - \dot{E}_{out}$$

where $\dot{E}_{in}$ is the energy injected at k_{source} and $\dot{E}_{out}$ is the energy dissipated at k_{sink}

Evidently to reach a stationary solution $\frac{dE}{dt} \stackrel{!}{=} 0$, meaning that $\dot{E}_{out} = \dot{E}_{in}$, i.e. energy has to dissipate at the same rate that it is injected into the cascade.

Energy flows along the cascade,
i.e. energy is injected at k_{source}
is transported from k_{source} all the
way to k_{small} via the cascade
and then dissipated at
k_{small}

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Now to describe the transport
of energy in k-space, we need
to look differentially at the distribution
of energy in k-space

$$\mathcal{E}(k,t) = \omega(k)\, n(k,t)$$

such that $E = \int d^d k\, \mathcal{E}(k,t)$
is the total energy

Based on the wave-kinetic equation we have

$$\frac{\partial}{\partial t} \mathcal{E}(k,t) = \omega(k)\, I[n](k,t) + \dot{E}_{in}(k) - \dot{E}_{out}(k)$$

where $\dot{E}_{in}(k)$ and $\dot{E}_{out}(k)$ characterize differentially the energy injection/dissipation at the source and sink, localized around k_{source} and respectively k_{sink}

So in the inertial range of wave-numbers

$$\frac{\partial}{\partial t} \mathcal{E}(k,t) = \omega(k)\, I[n](k,t)$$

which can be re-cast into the form of a continuity equation in k-space

$$\frac{\partial}{\partial t}\mathcal{E}(k,t) + \vec{\nabla}_k \vec{J}^k_{\mathcal{E}}(k,t) = 0$$

if we interpret the energy redistribution due to wave interactions, as the divergence of an energy current in k-space

$$\omega(k)\, I[n](k,t) = - \vec{\nabla}_k \vec{J}^k_{\mathcal{E}}(k,t)$$

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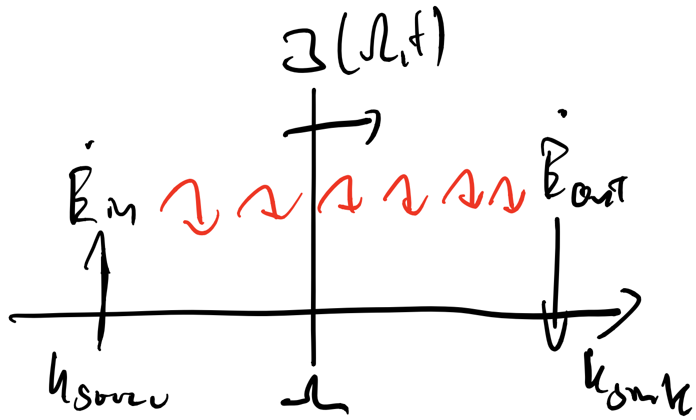
Now if we look at the change of the total energy up to some wave-number Ω



this is determined by the out flux

$J(k,t)$ of the energy

$$J(\Lambda,t) = -\frac{\partial}{\partial t}\int^{\Lambda} d^d k \; \varepsilon(k,t) = +\int^{\Lambda} d^d k \; \vec{\nabla} \vec{J}^{k}_{\varepsilon}(k,t)$$

$$= \underbrace{+\int_{\partial\Lambda} d\vec{\sigma}_k \; \vec{J}^{k}_{\varepsilon}(k,t)}_{\substack{\text{energy flux through} \\ \partial\Lambda \text{ a momentum shell}}}$$

which by virtue of the definition of the energy flux is given by

$$\boxed{J(\Lambda,t) = -\int^{\Lambda} d^d k \; \omega(k)\, I[n](k,t)}$$

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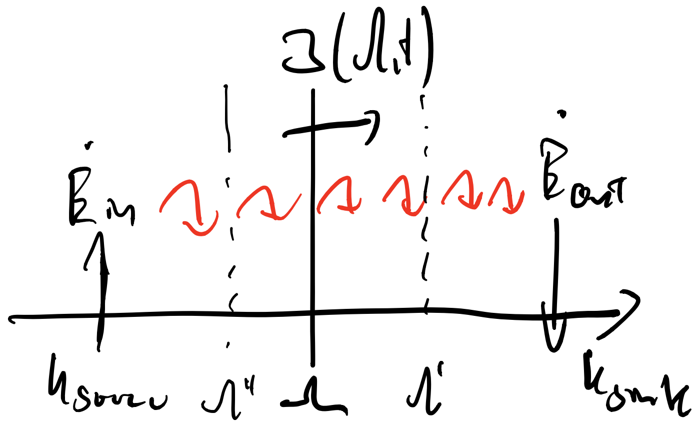
Now for stationary turbulent solutions the energy that is injected at k_{source} needs to be transported all the way to k_{sink}, without accumulating at any intermediate scale



So we obtain the condition that within the inertial range of momenta

$$J(\Lambda,t) = \dot{E}_{in} = \text{const}$$

i.e. the rate of energy transfer along the cascade is independent of Λ and equal to the energy injection rate $\dot{E}_{in}$

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Now if we determine the energy flux in the inertial range

$$\mathcal{J}(\Lambda, t) = -\int^{\Lambda} d^d k \; \omega(k) \; I[n_{k2}](k)$$

we can exploit scale invariance to re-write

$$= -\int^{\Lambda} d^d k \; \lambda^{-z} \omega(\lambda k) \; \lambda^{-\mu} I[n_{k2}](\lambda k)$$

Setting $\lambda = \frac{k_0}{k}$ to eliminate k dependence in ω and I, we get

$$= - \int d\Omega_k^{(a)} \int_0^{\Lambda} dk \, \underline{k^{d-1} \, k^{\mu+z}} \; \underline{k_0^{-(\mu+z)} \omega(k_0) \, I[n_{\mu z}](k_0)}$$

$\int dx \, x^{a-1} = \frac{x^a}{a}$

$$= - \Omega^{(d)} \Lambda^{d+\mu+z} \; \frac{k_0^{-(\mu+z)} \omega(k_0) \, I[n_{\mu z}](k_0)}{d+\mu+z}$$

We deduce that this is independent of the scale Λ iff

$$\boxed{d+\mu+z = 0}$$

which is precisely the same condition we had found to obtain stationary solutions $I[n](k)=0$ from Zakharov transformation and therefore provides an equivalent way to determine $S_0 = m+d$ for 3-wave interactions

Stated differently, the KZ spectra are associated with a scale (k) independent energy flux, which is needed to realize the cascade

We again note on the basis of this (much) simpler derivation that the exponent s_0 of the KZ spectrum is <u>universal</u> in the sense that it is insensitive to the details of the underlying microscopic theory and <u>only</u> <u>sensitive</u> to the <u>scaling properties</u> <u>of interactions</u> (m) <u>and dimensionality</u> (d) of the system //

Now in actually evaluating the energy flux, there is a subtlety as

$$\mathcal{J}(k,t) = -\Omega^{(d)} \frac{k_0^{-(\mu+2)} \omega(k_0) \, I[n_{KZ}](k_0)}{d+\mu+z}$$

as both $I[n_{KZ}](k_0) = 0$ and $d+\mu+z = 0$ for the KZ spectrum.

Even though $I[n_{KZ}](k_0) = 0$ as the solution is stationary, the energy flux $\mathcal{J}(k,t)$ is neither 0 nor ∞ but has a liftable singularity in the limit $d+\mu+z \to 0^+$

$$J_S = \lim_{d+\mu+z \to 0^+} J(k,t)$$

$$= -\Omega^{(d)} k_0^d \,\omega(k_0) \lim_{\mu+d+z \to 0} \frac{I[n_{KZ}](k_0)}{d+\mu+z}$$

$\Downarrow$

Now using the form of $I[n_{KZ}](k)$ determined via Zakharov transformations

$$I[n_{KZ}](k) = \frac{1}{2\Omega^{(d)}} \int d\Omega_k \int d\Omega_1 \int d\Omega_2 \int dk_1\, k_1^{d-1} \int dk_2\, k_2^{d-1}$$
$$R(\vec{k}|\vec{k}_1,\vec{k}_2) \left[1 - \left|\frac{\vec{k}}{\vec{k}_1}\right|^{\mu+d} - \left|\frac{\vec{k}}{\vec{k}_2}\right|^{\mu+d} \right]$$

We can re-express

$$\left|\frac{\vec{k}}{\vec{k}_1}\right|^{M+d} = \left|\frac{\vec{k}}{\vec{k}_1}\right|^{M+d+z}\left|\frac{\vec{k}_1}{\vec{k}}\right|^{z} = \left|\frac{\vec{k}}{\vec{k}_1}\right|^{M+d+z}\frac{\omega(\vec{k}_1)}{\omega(\vec{k})}$$

and expand $|X|^{\varepsilon} = 1 + \varepsilon \log(|X|)$

and the leading term

$$I[n_k n_2](k) = \frac{1}{2\Omega^{(d)}}\int d\Omega_k \int d\Omega_1 \int d\Omega_2 \int dk_1\, k_1^{d-1}\int dk_2\, k_2^{d-1}$$

$$\frac{R(\vec{k}(\vec{k}_1,\vec{k}_2))}{\omega(\vec{k})}\left[\omega(\vec{k}) - \omega(\vec{k}_1) - \omega(\vec{k}_2)\right] = 0$$

vanishes due to energy conservation

Conversely the ~~energy flux~~ ζ_S is determined by the next term in the expansion

$$I[nn_2](k) = \frac{1}{2\Omega^{(d)}} \int d\Omega_k \int d\Omega_1 \int d\Omega_2 \int dk_1\, k_1^{d-1} \int dk_2\, k_2^{d-1}$$

$$(d+\mu+\epsilon)\, \frac{R(\vec{k}|\vec{k}_1,\vec{k}_2)}{\omega(\vec{k})} \left[\omega(\vec{k}) \log(1) - \omega(\vec{k}_1) \log\left(\left|\frac{\vec{k}}{\vec{k}_1}\right|\right) - \omega(\vec{k}_2) \log\left(\left|\frac{\vec{k}}{\vec{k}_2}\right|\right) \right]$$

So the energy flux

$$J_S = \frac{\omega(k_0) k_0^d}{2} \int d\Omega_{k_0} \int d\Omega_1 \int d\Omega_2 \int dk_1\, k_1^{d-1} \int dk_2\, k_2^{d-1}$$

$$R(\vec{k}_0|\vec{k}_1,\vec{k}_2) \left[\left|\frac{\vec{k}_1}{\vec{k}_0}\right|^2 \log\left(\left|\frac{\vec{k}_0}{\vec{k}_1}\right|\right) + \left|\frac{\vec{k}_2}{\vec{k}_0}\right|^2 \log\left(\left|\frac{\vec{k}_0}{\vec{k}_2}\right|\right) \right]$$

is manifestly finite. Note that while we have performed the analysis for the energy flux in 3-wave kinetic equation, analogous analyses can be performed for other conserved quantities in different interactions

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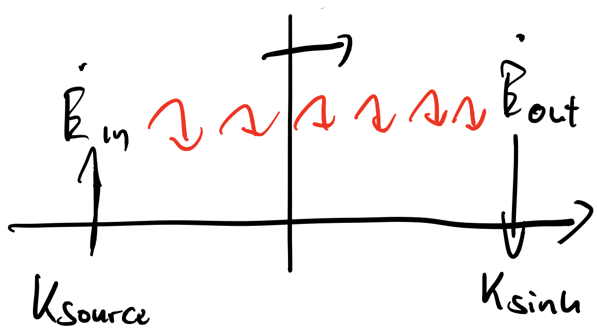
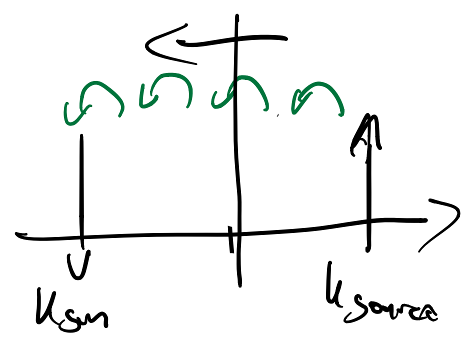
Based on the sign of ζ_s, we can also determine the direction of the energy flux (i.e. from small to large wave numbers or vice versa) to distinguish

Direct cascade ($\zeta_s > 0$)



Inverse cascade ($\zeta_s < 0$)



By equating $|\zeta_s| = \dot{E}_{in}$, one can further determine the non-universal amplitude (n_0) as a function of the energy injection rate ($\dot{E}_{in}$) (c.f. exercises)