

Hydrodynamics from Boltzmann equation

We discussed that BTE describes non-equilibrium dynamics of a system on time and distance scales, much larger than microscopic interaction length τ_0 and collision time τ_c

We usually expect that on very large time and distance scales, the system should relax towards equilibrium and become amenable to near-equilibrium description in terms of hydrodynamics, as developed in Chapter I.

Now if we compare the two descriptions

Kinetic theory	Hydrodynamics
Phase-space distribution $f(t, \vec{r}, \vec{p})$	Densities of (conserved) quantities $n(t, \vec{r}), e(t, \vec{r}), \vec{v}(t, \vec{r})$
Boltzmann equation $\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \vec{\nabla}_r + \vec{F} \vec{\nabla}_p\right) f = C[f]$	Conservation laws $\frac{\partial}{\partial t} X + \vec{\nabla}_r \cdot \vec{J}_X = 0$
	Constitutive relations $\vec{J}_X = \vec{J}_X^0 + L_{XY} \vec{F}_Y$
	Equations of state $X = X(T, \mu, \vec{v})$

we clearly see that hydrodynamics involves much fewer dynamical variables,

$$n(t, \vec{r}) = \int_{\vec{p}} f(t, \vec{r}, \vec{p}), \dots$$

so we need to understand how (under what circumstances) such a simplification is possible

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We consider the evolution of expectation values of one-body operator $\chi(t, \vec{r}, \vec{p})$

$$\langle \chi(t, \vec{r}, \vec{p}) \rangle_{\vec{p}} = \frac{1}{n(t, \vec{r})} \int_{\vec{p}} \chi(t, \vec{r}, \vec{p}) f(t, \vec{r}, \vec{p})$$

for which we can derive BOMs by taking the appropriate moments of the Boltzmann equation (c.f. previous lectures/exercises)

$$\begin{aligned} \frac{d}{dt} n \langle \chi \rangle_{\vec{p}} &= n \left\langle \frac{\partial \chi}{\partial t} \right\rangle_{\vec{p}} - \int_{\vec{p}} \chi \frac{\vec{p}}{m} \vec{\nabla}_{\vec{r}} f \\ &\quad - \int_{\vec{p}} \chi \vec{F} \vec{\nabla}_{\vec{p}} f + \int_{\vec{p}} \chi C(f) \end{aligned}$$

Specifically if $\chi(t, \vec{r}, \vec{p})$ is a collision invariant

$$\int_{\vec{p}} \chi C(f) = n \langle \chi C(f) \rangle_{\vec{p}} = 0$$

further re-writing

$$\int_{\vec{p}} \chi \vec{E} \cdot \vec{v}_r f = \vec{v}_r \cdot n \langle \chi \vec{E} \rangle_{\vec{p}} - n \langle \vec{E} \cdot \vec{v}_r \chi \rangle_{\vec{p}}$$

$$\int_{\vec{p}} \chi \vec{v}_p \cdot \vec{v}_p f = -n \vec{v}_p \cdot \langle \vec{v}_p \chi \rangle_{\vec{p}}$$

we get

$$\begin{aligned} \frac{d}{dt} n \langle \chi \rangle_{\vec{p}} - n \left\langle \frac{d\chi}{dt} \right\rangle_{\vec{p}} + \vec{v}_r \cdot n \langle \chi \vec{E} \rangle_{\vec{p}} - n \langle \vec{E} \cdot \vec{v}_r \chi \rangle_{\vec{p}} \\ (*) \qquad \qquad \qquad = n \vec{v}_p \cdot \langle \vec{v}_p \chi \rangle_{\vec{p}} \end{aligned}$$

Based on which we can now obtain all the relevant forms of the balance equations for fluid dynamics

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mass density: $\rho(t, \vec{r}) = m n(t, \vec{r})$

$$\boxed{\chi = m} \quad \parallel \quad \frac{d}{dt} \chi = 0 \quad \vec{\nabla}_r \chi = 0 \quad \vec{\nabla}_p \chi = 0$$

$$\frac{d}{dt} \rho(t, \vec{r}) + \vec{\nabla}_r \rho(t, \vec{r}) \left\langle \frac{\vec{p}}{m} \right\rangle_p = 0$$

Density $\boxed{\vec{v}(t, \vec{r}) = \left\langle \frac{\vec{p}}{m} \right\rangle_p}$ we got the
usual continuity equation

$$\boxed{\frac{d}{dt} \rho(t, \vec{r}) + \vec{\nabla}_r (\rho(t, \vec{r}) \vec{v}(t, \vec{r})) = 0}$$

momentum density:

$$\chi^i = p^i \quad \parallel \quad \frac{d}{dt} \chi = 0 \quad \vec{\nabla}_r \chi = 0 \quad \frac{d}{dp_j} \chi^i = \delta^{ij}$$

$$\frac{d}{dt} \rho(t, \vec{r}) v^i(t, \vec{r}) + \frac{d}{dr^j} \rho(t, \vec{r}) \left\langle \frac{p^i p^j}{m} \right\rangle_p = \frac{\rho(t, \vec{r})}{m} F^i(t, \vec{r})$$

which now involves a new moment $\left\langle \frac{p^i p^j}{m} \right\rangle$

If the system is locally moving with velocity $\vec{v}(t, \vec{r}) = \langle \frac{\vec{p}}{m} \rangle_{\vec{p}}$ it is useful to separate the internal motion $\frac{\vec{p}}{m} - \vec{v}$ from the average motion

Denoting the stress tensor Π^{ij} as

$$\Pi^{ij}(t, \vec{r}) = \rho(t, \vec{r}) \left\langle \left(\frac{\vec{p}}{m} - \vec{v}(t, \vec{r}) \right)^i \left(\frac{\vec{p}}{m} - \vec{v}(t, \vec{r}) \right)^j \right\rangle_{\vec{p}}$$

we have

$$\begin{aligned} \Pi^{ij} &= \rho \left\langle \left(\frac{\vec{p}}{m} - \vec{v} \right)^i \left(\frac{\vec{p}}{m} - \vec{v} \right)^j \right\rangle_{\vec{p}} \\ &= \rho \left\langle \frac{p^i}{m} \frac{p^j}{m} \right\rangle_{\vec{p}} - \rho \underbrace{\left\langle \frac{p^i}{m} v^j \right\rangle_{\vec{p}}}_{= v^i v^j} - \rho \underbrace{\left\langle v^i \frac{p^j}{m} \right\rangle_{\vec{p}}}_{= v^i v^j} + \rho \left\langle v^i v^j \right\rangle_{\vec{p}} \\ &= \rho \left\langle \frac{p^i}{m} \frac{p^j}{m} \right\rangle_{\vec{p}} - \rho v^i v^j \end{aligned}$$

such that

$$\rho \left\langle \frac{p^i}{m} \frac{p^j}{m} \right\rangle_{\vec{p}} = \underbrace{\Pi^{ij}(t, \vec{r})}_{\text{internal motion of particles}} + \rho \underbrace{v^i(t, \vec{r}) v^j(t, \vec{r})}_{\text{average motion of local cell}}$$

Now collecting everything, we get the familiar balance equation

$$\frac{d}{dt} \rho(t, \vec{r}) v^i(t, \vec{r}) + \frac{d}{dr^j} \left(\rho(t, \vec{r}) v^i(t, \vec{r}) v^j(t, \vec{r}) + \pi^{ij}(t, \vec{r}) \right) = \frac{\rho(t, \vec{r})}{m} F^i(t, \vec{r})$$

energy density:

We consider the internal energy density e , related to the internal motion of particles in a gas

$$\chi(t, \vec{r}, \vec{p}) = \frac{(\vec{p} - m\vec{v}(t, \vec{r}))^2}{2m}$$

$$\frac{d}{dt} \chi(t, \vec{r}, \vec{p}) = - \frac{d\vec{v}}{dt} (\vec{p} - m\vec{v}(t, \vec{r}))$$

$$\frac{d}{dr^j} \chi(t, \vec{r}, \vec{p}) = - \frac{dv^i}{dr^j} (\vec{p} - m\vec{v}(t, \vec{r}))^i$$

$$\frac{d}{dp_i} \chi(t, \vec{r}, \vec{p}) = \left(\frac{\vec{p}}{m} - \vec{v}(t, \vec{r}) \right)^i$$

So defining the orbital energy density

$$e(t, \vec{r}) = n \langle \mathcal{K} \rangle_{\vec{r}} = \frac{1}{2} \rho(t, \vec{r}) \left\langle \left(\frac{\vec{p}}{m} - \vec{v}(t, \vec{r}) \right)^2 \right\rangle_{\vec{r}}$$

We can evaluate again (*) with

$$\left\langle \frac{d\mathcal{K}}{dt} \right\rangle_{\vec{r}} = - \frac{d\vec{v}}{dt} \langle \vec{p} - m\vec{v}(t, \vec{r}) \rangle_{\vec{r}} = 0$$

$$\begin{aligned} n \langle \mathcal{K} \frac{\vec{p}}{m} \rangle_{\vec{r}} &= \frac{1}{2} \rho(t, \vec{r}) \left\langle \left(\frac{\vec{p}}{m} - \vec{v}(t, \vec{r}) \right)^2 \frac{\vec{p}}{m} \right\rangle_{\vec{r}} \\ &= \frac{1}{2} \rho(t, \vec{r}) \left\langle \left(\frac{\vec{p}}{m} - \vec{v}(t, \vec{r}) \right)^2 \left(\frac{\vec{p}}{m} - \vec{v}(t, \vec{r}) \right) \right\rangle_{\vec{r}} \\ &\quad + e(t, \vec{r}) \vec{v}(t, \vec{r}) \end{aligned}$$

where the first term describes the energy flux in LRF

$$\vec{J}_u(t, \vec{r}) = \frac{1}{2} \rho(t, \vec{r}) \left\langle \left(\frac{\vec{p}}{m} - \vec{v}(t, \vec{r}) \right)^2 \left(\frac{\vec{p}}{m} - \vec{v}(t, \vec{r}) \right) \right\rangle_{\vec{r}}$$

$$n \langle \mathcal{K} \frac{\vec{p}}{m} \rangle_{\vec{r}} = \vec{J}_u(t, \vec{r}) + e(t, \vec{r}) \vec{v}(t, \vec{r})$$

$$\eta \left\langle \frac{\vec{p}}{m} \vec{\nabla}_r \chi \right\rangle_{\vec{p}} = -\rho \left\langle \frac{\vec{p}^j}{m} \frac{\partial v^i}{\partial r^k} \left(\frac{\vec{p}}{m} - v \right)^i \right\rangle_{\vec{p}}$$

$$\text{inserting } \left\langle v^i \left(\frac{\vec{p}}{m} - v \right)^i \right\rangle_{\vec{p}} = 0$$

$$= -\rho \left\langle \left(\frac{\vec{p}}{m} - v \right)^i \left(\frac{\vec{p}}{m} - v \right)^i \right\rangle_{\vec{p}} \frac{\partial v^i}{\partial r^j}$$

$$= -\Pi^{ij} \frac{\partial v^i}{\partial r^j}$$

$$\left\langle \vec{\nabla}_p \chi \right\rangle_{\vec{p}} = \left\langle \frac{\vec{p}}{m} - \vec{v}(t, \vec{r}) \right\rangle_{\vec{p}} = 0$$

Now collecting everything we get
the balance equation for the internal
energy

$$\frac{\partial}{\partial t} e(t, \vec{r}) + \vec{\nabla}_r \cdot \left(\vec{J}_q(t, \vec{r}) + e(t, \vec{r}) \vec{v}(t, \vec{r}) \right) = -\Pi^{ij}(t, \vec{r}) \frac{\partial v^i}{\partial r^j}(t, \vec{r})$$

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So in summary we have now rederived the equations of motion for ρ, \vec{v}, e from chapter I

$$\frac{d}{dt} \rho(t, \vec{r}) + \vec{\nabla}_r \rho(t, \vec{r}) \vec{v}(t, \vec{r}) = 0$$

$$\frac{d}{dt} \rho(t, \vec{r}) v^i(t, \vec{r}) + \frac{d}{dr^j} \left(\rho(t, \vec{r}) v^i(t, \vec{r}) v^j(t, \vec{r}) + \Pi^{ij}(t, \vec{r}) \right) = \frac{\rho(t, \vec{r})}{m} F^i(t, \vec{r})$$

$$\frac{d}{dt} e(t, \vec{r}) + \vec{\nabla}_r \left(\vec{J}_a(t, \vec{r}) + e(t, \vec{r}) \vec{v}(t, \vec{r}) \right) = -\Pi^{ij}(t, \vec{r}) \frac{\partial v^i}{\partial r^j}(t, \vec{r})$$

along with a set of equations that relate the various microscopic quantities ρ, \vec{v}, e as well as Π, \vec{J}_a to moments of the single particle phase space distribution

$$\rho(t, \vec{r}) = m \int_{\vec{p}} \psi(t, \vec{r}, \vec{p})$$

$$\vec{V}(t, \vec{r}) = \frac{1}{\rho(t, \vec{r})} \int_{\vec{p}} \vec{p} \psi(t, \vec{r}, \vec{p})$$

$$e(t, \vec{r}) = \int_{\vec{p}} \frac{(\vec{p} - m\vec{V}(t, \vec{r}))^2}{2m} \psi(t, \vec{r}, \vec{p})$$

$$\vec{J}_a(t, \vec{r}) = \int_{\vec{p}} \frac{(\vec{p} - m\vec{V}(t, \vec{r}))^2}{2m} \left(\frac{\vec{p}}{m} - \vec{V}(t, \vec{r}) \right) \psi(t, \vec{r}, \vec{p})$$

$$\Pi^{is}(t, \vec{r}) = \int_{\vec{p}} \frac{(\vec{p} - m\vec{V})^i (\vec{p} - m\vec{V})^s}{m} \psi(t, \vec{r}, \vec{p})$$

Now to obtain a closed set of hydrodynamic evolution equations, one needs to find approximate solutions for $\psi(t, \vec{r}, \vec{p})$ and calculate these moments to obtain equations of state and constitutive equations

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Generally there are different ways to truncate the Boltzmann equation, in order to obtain a closed set of evolution equations for its moments

→ active topic of research in particular for statistical hydrodynamics

Ultimately in all hydrodynamic flows the dynamics is described in terms of local thermodynamic variables and their space/time derivatives, so the truncation of the Boltzmann equation leads to a reduction of the degrees of freedom

We will discuss the traditional Chapman-Enskog expansion based on a strict expansion in the Knudsen number

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Chapman Enskog expansion

We expand the solution to the Boltzmann equation in powers of the Knudsen number $Kn \ll 1$

$$f(t, \vec{r}, \vec{p}) = \underline{f^{(0)}(t, \vec{r}, \vec{p})} + f^{(1)}(t, \vec{r}, \vec{p}) + \dots$$

where at lowest order we will recover the local equilibrium solutions

$$f^{(0)}(t, \vec{r}, \vec{p}) = f^{(0)}(T(t, \vec{r}), n(t, \vec{r}), \vec{v}(t, \vec{r}), \vec{p})$$

such that derivatives w.r.t. t and \vec{r} can be expressed in terms of thermodynamic variables, which are obtained via the matching conditions

$$n(t, \vec{r}) = \int_{\vec{p}} f(t, \vec{r}, \vec{p})$$

$$\vec{v}(t, \vec{r}) = \frac{1}{n(t, \vec{r})} \int_{\vec{p}} \frac{\vec{p}}{m} f(t, \vec{r}, \vec{p})$$

$$\frac{3}{2} n(\vec{r}, \vec{r}) k_B T(\vec{r}, \vec{r}) = \int_{\vec{p}} \frac{(\vec{p} - m\vec{v}(\vec{r}, \vec{r}))^2}{2m} f(\vec{r}, \vec{r}, \vec{p})$$

Now in order to develop an expansion where $f^{(1)}, f^{(2)}, \dots$ can be truncated at some finite order, we need to assume that they do not carry any conserved quantities, i.e. the full information about the local thermodynamic properties n, \vec{v}, T is contained in the local order $f^{(0)}$ such that

$$n(\vec{r}, \vec{r}) = \int_{\vec{p}} f(\vec{r}, \vec{r}, \vec{p}) \stackrel{!}{=} \int_{\vec{p}} \underline{f^{(0)}(\vec{r}, \vec{r}, \vec{p})}$$

and similarly for \vec{v}, T .

Nevertheless this expansion is not guaranteed to converge, but the low order truncations can still be useful, and we will consider 0th and 1st order

0th order: Ideal fluid dynamics

$$O(kn^0) \quad C[f^{(0)}] = 0$$

$\Rightarrow f^{(0)}$ is local equilibrium solution

$$f^{(0)}_{(t,\vec{r})} = n(t,\vec{r}) \left(\frac{2\pi t^2}{m k_B T(t,\vec{r})} \right)^{3/2} \exp\left(- \frac{(\vec{p} - m\vec{v}(t,\vec{r}))^2}{2m k_B T(t,\vec{r})} \right)$$

Where by virtue of the matching $n(t,\vec{r})$ and $\vec{v}(t,\vec{r})$ are the same density and velocity as in the equations of motion

Based on the explicit form of $f^{(0)}$ we can then calculate the leading contributions in the expansion of the energy flux

$$\vec{z}_u = \vec{z}_u^{(0)} + \vec{z}_u^{(1)} + \dots$$

and stress tensor

$$\Pi^{ij} = \Pi_{(0)}^{ij} + \Pi_{(1)}^{ij} + \dots$$

as

$$\vec{z}_u^{(0)}(t, \vec{r}) = \int_{\vec{p}} \frac{(\vec{p} - m\vec{v}(t, \vec{r}))^2}{2m} \left(\frac{\vec{p}}{m} - \vec{v}(t, \vec{r}) \right) f_{(0)}^{(0)}(t, \vec{r}, \vec{p})$$

$$\Pi_{(0)}^{ij}(t, \vec{r}) = \int_{\vec{p}} \frac{(\vec{p} - m\vec{v}(t, \vec{r}))^i (\vec{p} - m\vec{v}(t, \vec{r}))^j}{m} f_{(0)}^{(0)}(t, \vec{r}, \vec{p})$$

yields (c.f. exercises)

$$\vec{z}_u^{(0)}(t, \vec{r}) = 0$$

$$\Pi_{(0)}^{ij}(t, \vec{r}) = P(t, \vec{r}) \delta^{ij}$$

where $P(t, \vec{r}) = n(t, \vec{r}) k_B T(t, \vec{r})$

is the pressure of an ideal gas

By using this rule the evolution equations

$$\frac{d}{dt} \rho(t, \vec{r}) + \vec{\nabla}_r \rho(t, \vec{r}) \vec{v}(t, \vec{r}) = 0$$

$$\frac{d}{dt} \rho(t, \vec{r}) v^i(t, \vec{r}) + \frac{d}{dr^j} \left(\rho(t, \vec{r}) v^i(t, \vec{r}) v^j(t, \vec{r}) + \pi^{ij}(t, \vec{r}) \right) = \frac{\rho(t, \vec{r}) F^i(t, \vec{r})}{m}$$

$$\frac{d}{dt} e(t, \vec{r}) + \vec{\nabla}_r \left(\vec{J}_a(t, \vec{r}) + e(t, \vec{r}) \vec{v}(t, \vec{r}) \right) = -\pi^{ij}(t, \vec{r}) \frac{\partial v^i}{\partial r^j}(t, \vec{r})$$

we obtain

$$\frac{d}{dt} \rho + \vec{\nabla}(\rho \vec{v}) = 0$$

$$\rho \frac{d}{dt} \vec{v} + \rho(\vec{v} \cdot \vec{\nabla}) \vec{v} = -\vec{\nabla} P + \frac{\rho}{m} \vec{F}$$

$$\frac{d}{dt} e + \vec{\nabla}(e \vec{v}) = -P(\vec{\nabla} \vec{v})$$

which are the equations of motion for an ideal fluid.

1st order: Navier-Stokes Hydrodynamics

Next order in $\hbar v_F$, the Boltzmann equation takes the form

$$O(\hbar v_F): \left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \vec{\nabla}_r + \vec{F} \vec{\nabla}_p \right) f^{(0)} = \mathcal{S}\mathcal{C}[f^{(0)}, f^{(1)}]$$

and we will focus on the relaxation time approximation

$$\mathcal{S}\mathcal{C}[f^{(0)}, f^{(1)}] = -\frac{1}{\tau_R} f^{(1)}, \text{ such}$$

that

$$f^{(1)} = -\tau_R \left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \vec{\nabla}_r + \vec{F} \vec{\nabla}_p \right) f^{(0)}$$

Now in order to evaluate $f^{(1)}$ explicitly and calculate its moments, we need to evaluate

e.g.

$$\frac{\partial}{\partial t} f^{(0)} = \frac{\partial n}{\partial t} \frac{\partial f^{(0)}}{\partial n} + \frac{\partial T}{\partial t} \frac{\partial f^{(0)}}{\partial T} + \frac{\partial v^i}{\partial t} \frac{\partial f^{(0)}}{\partial v^i}$$

which will involve time derivatives
of the local thermodynamic
variables $\frac{dn}{dt}$, $\frac{dT}{dt}$, $\frac{d\vec{v}}{dt}$.

However to leading order in kn , these
satisfy the ideal hydrodynamic
equations (up to corrections of order kn),
which we can use to express
time derivatives in terms of
spatial derivatives, e.g.

$$\frac{dn}{dt} + \vec{\nabla} \cdot (n\vec{v}) = 0 \Rightarrow \boxed{\frac{dn}{dt} = -\vec{\nabla} \cdot (n\vec{v})}$$

continuity equation

and similarly for $\frac{dT}{dt}$ and $\frac{dv^i}{dt}$

Eliminating the time derivatives in favor
of spatial derivatives and
density $\vec{u}_p = \frac{\vec{p}}{m} - \vec{v}(t, \vec{r})$ one finds

(c). exercises of chapter 7.5 of Pethöcz

$$f^{(1)} = -\bar{c}_R \left[\frac{1}{T} (\vec{u}_p \cdot \vec{\nabla} T) \left(\frac{m \vec{u}_p^2}{2k_B T} - \frac{5}{2} \right) + \left(\frac{\partial V_n}{\partial x_e} + \frac{\partial V_e}{\partial x_n} \right) \frac{m}{2k_B T} \left(u_p^k u_p^k - \frac{1}{3} \delta_{ij} u_p^i u_p^j \right) \right]$$

induced dissipative corrections due
to temperature and velocity gradients

Based on $f^{(1)}$ one can then calculate the non-equilibrium (dissipative) corrections to the energy flux $\vec{J}_E^{(1)}$ and stress tensor $\Pi_{(1)}^{ij}$ as

$$\begin{aligned}\Pi_{(1)}^{ij} &= \int_{\vec{p}} \frac{(\vec{p} - m\vec{v}(t, \vec{r}))^i (\vec{p} - m\vec{v}(t, \vec{r}))^j}{m} f^{(1)}(t, \vec{r}, \vec{p}) \\ &= m^4 \int_{\vec{u}_p} u_i u_j f^{(1)}(t, \vec{r}, \vec{p})\end{aligned}$$

Since the term $f^{(1)}$ & $\vec{u}_p \cdot \vec{\nabla} T$ is odd in u it will vanish under integration

$$= -\frac{m^5 \bar{E}_R}{2k_B T} \left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} \right) \int_{\vec{u}_p} u_i u_j \left(u_k u_l - \frac{1}{3} \delta_{kl} u_p^2 \right) f^{(2)}$$

Evaluating the integral, one finds

$$\int_{\vec{u}_p} u_i u_j \left(u_k u_l - \frac{1}{3} \delta^{kl} u_p^2 \right) f^{(0)} = \frac{2n (k_B T)^2}{m^5} \left(\frac{\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}}{2} - \frac{1}{3} \delta^{ij} \delta^{kl} \right)$$

such that

$$\begin{aligned} \Pi_{ij}^{(1)} &= -n k_B T \tau_R \left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} \right) \left(\frac{\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}}{2} - \frac{1}{3} \delta^{ij} \delta^{kl} \right) \\ &= -n k_B T \tau_R \underbrace{\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta^{ij} \nabla \cdot \vec{v} \right)}_{= 2 \sigma^{ij}} \end{aligned}$$

so we find

$$\boxed{\Pi_{ij}^{(1)} = -2\eta \sigma^{ij}}$$

$$\boxed{\eta = n k_B T \tau_R}$$

with shear viscosity microscopically calculated from Chapman-Enskog expansion

Similarly one can show that the first non-equilibrium correction to the heat flux is given by (c.f. exercises)

$$\vec{J}_u^{(1)} = -\kappa \vec{\nabla} T$$

with

$$\kappa = \frac{5}{2} n \frac{k_B^2 T}{m}$$

Since the form of the constitutive relations is identical to the ones discussed in Chapter I, this leads to the Navier-Stokes equations



Based on the microscopic calculations of the transport coefficients η, κ we can also gain some better understanding of what affects the fluid dynamical properties of a given material

We see that

$\eta, \kappa \propto \tau R$

noticing that

Smaller relaxation time \rightarrow smaller distances for local equilibrium

more ideal fluid \leftarrow smaller viscosity

If we further estimate

$$\bar{c}_R \sim \bar{c}_{mp} \sim \frac{1}{n c_{tot} \langle v \rangle}$$

$$\text{with } \langle v \rangle \sim \sqrt{\frac{k_B T}{m}}$$

$$\Rightarrow \eta = n k_B T \bar{c}_R \sim \frac{\sqrt{m k_B T}}{c_{tot}}$$

which is approximately independent of the density, but inversely proportional to the total cross section

Stronger interactions \rightarrow smaller domains
for local equilibrium
 \downarrow
more ideal fluid \leftarrow smaller viscosity