Solutions to Boltzmann equation

Global equilibrium $\left(\frac{\partial f_{eq}}{\partial t} = 0 \right)$

We found that in the absence of external forces

Maxwell-Boltzmann distribution

$$f(t,\vec{r},\vec{p}) = f_{eq}(\vec{p}) = n \left(\frac{2\pi\hbar^2}{mk_0T} \right)^{3/2} e^{-\frac{(\vec{p}-m\vec{V}_0)^2}{2mk_0T}}$$

is a stationary solution of the Boltzmann equation, described by

n number density

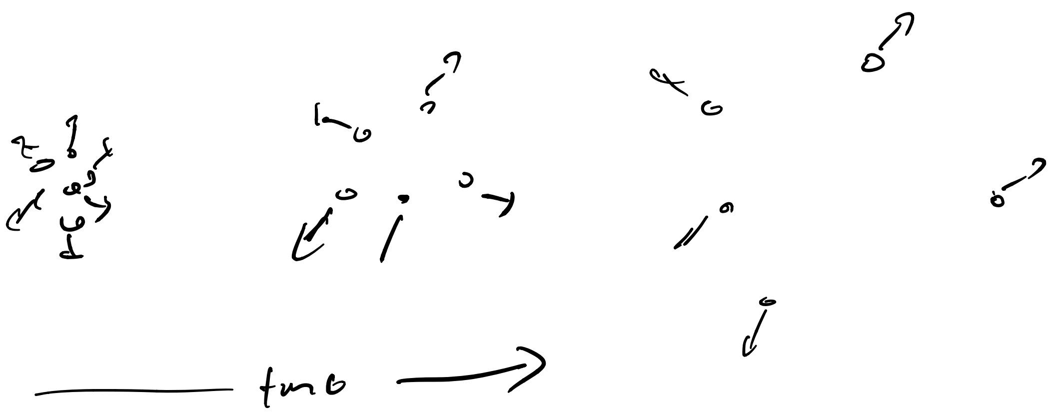
$\vec{V}_0$ velocity

T temperature

Since this distribution describes a thermodynamic equilibrium state, its entropy is maximal and one could expect any system to approach such a state on large time scales as $\frac{dS}{dt} \geq 0$ in the Boltzmann equation

However there are counterexamples and the H-theorem does not guarantee convergence to an equilibrium state at late times

e.g. If phase space is unbounded the system may continue to expand



Systems confined to a finite volume may exhibit periodic behavior (c.f. exercise sheet 8)

Evidently global equilibrium solutions are relatively easy to find but often not particularly interesting, and we will now try to construct approximate solutions exploiting once again the fact that in many systems there can be a separation of scales

<u>Local equilibrium solutions:</u> $\left(f^{(0)}(t,\vec{r},\vec{p})\right)$

Strictly speaking are not solutions of BTE, but only defined by the requirement that

$$C[f^{(0)}](t,\vec{r},\vec{p})=0$$

Nevertheless conceptually important if we assume a separation of scales

Deriving the BTE we had already encountered different scales

fast

τ_c microscopic time scale for single collision (not described by BTE)

slow

τ_s, τ_e system responds to gradients and external forces

Denoting $\tau_s \sim \left(\vec{v}\,\frac{\nabla f}{f}\right)^{-1} \sim \frac{L}{v_{th}}$ with a macroscopic length scale L we should also compare τ_e to the typical time scale between collision events mean free time:

$$\tau_{mfp} \sim \left(\frac{\left(\frac{\delta f}{\delta t}\right)_{coll}}{f} \right)^{-1} \sim \frac{1}{n \sigma_{tot} v_{th}}$$

Now using $\tau_{mfp} = \frac{l_{mfp}}{v_{th}}$ we get

mean free path:

$$l_{mfp} \sim \frac{1}{n \sigma_{tot}}$$

Such that the ratio of the scales

$$\frac{\tau_{mfp}}{\tau_S} \sim \frac{l_{mfp}}{L} \equiv Kn$$

Now it is important to remember that the derivation of the Boltzmann equation only requires collision time $\tau_c \ll \tau_{mfp}$ typical time between collisions

but so far we did not make any assumptions on the ordering of the scales $\tau_{mfp}, \tau_s, \tau_e$

Now if we assume a separation of scales between τ_{mfp} and τ_s, τ_e we can distinguish two qualitatively different regimes, where we can systematically derive solutions to BTE

$Kn \gg 1$

$\tau_{mfp} \gg \tau_s, \tau_e$

fast

microscopic — τ_c

BTE — τ_s, τ_e

slow — τ_{mfp}

System is essentially free streaming up to corrections due to rare collision events

$Kn \ll 1$

$\tau_{mfp} \ll \tau_s, \tau_e$

fast

microscopic — τ_c

BTE — τ_{mfp}

slow — τ_s, τ_e

Several collisions occur during the response to gradients and external forces

When $\boxed{Kn \ll 1}$ collisions are very frequent on macroscopic time scales τ_s, τ_e

Since collisions are local events that lead to equilibration of the momentum distribution, we can assume that locally the system relaxes to an equilibrium momentum distribution long before it eventually may or may not reach global equilibrium

Becomes useful to characterize intermediate states by

local equilibrium solution: $\left(C[f^{(0)}] = 0 \right)$

$$f^{(0)}(t,\vec{r},\vec{p}) = n(t,\vec{r}) \left(\frac{2\pi\hbar^2}{m k_B T(t,\vec{r})}\right)^{3/2} e^{-\frac{(\vec{p}-m\vec{V}(t,\vec{r}))}{2m\, k_B T(t,\vec{r})}}$$

where $n(t,\vec{r})$, $T(t,\vec{r})$, $\vec{V}(t,\vec{r})$ are locally defined at each point in space in time

//

Now if we consider the limit $Kn \ll 1$ we can obtain approximate solutions to BTE by expanding around local equilibrium distribution

$$f(t,\vec{r},\vec{p}) = \underbrace{f^{(0)}(t,\vec{r},\vec{p})}_{\text{local equilibrium}} + \underbrace{f^{(1)}(t,\vec{r},\vec{p})}_{\text{non-equilibrium corrections}} + \cdots$$

where $f^{(1)}(t,\vec{r},\vec{p})$ is a small correction with $|f^{(1)}(t,\vec{r},\vec{p})| \sim O(Kn)$

Expanding in Kn, the dynamics of $f^{(1)}$ will be governed by linearized BTE and there are different strategies to organize the expansion based on the power counting of residual terms in BTE

We will discuss one particular expansion scheme which we will need to derive hydrodynamics from BTE

Hilbert expansion:

Generally we find that based on our ansatz

$$\frac{\partial f^{(0)}}{\partial t} + \frac{\partial f^{(1)}}{\partial t} + \vec{v}\vec{\partial}_r f^{(0)} + \vec{v}\vec{\partial}_r f^{(1)} + \vec{F}\vec{\partial}_p f^{(0)} + \vec{F}\vec{\partial}_p f^{(1)} = C[f^{(0)}+f^{(1)}]$$

where we can express the collision integral as

$$C[f^{(0)}+f^{(1)}] = C[f^{(0)}] + \delta C[f^{(0)}, f^{(1)}]$$

with

$$\delta C[f^{(0)}, f^{(1)}] = \frac{1}{2}\int_{p_2 p_3 p_4} \tilde{w}(p_1 p_2 \to p_3 p_4)$$

$$\left[f_3^{(0)} f_4^{(1)} + f_3^{(1)} f_4^{(0)} - f_1^{(0)} f_2^{(1)} - f_1^{(1)} f_2^{(0)}\right]$$

So if we powercount the individual terms, we have

$$C[f^{(0)}] \sim \frac{1}{\tau_{mfp}} f^{(0)}$$

$$\delta C[f^{(0)}, f^{(1)}] \sim \frac{1}{\tau_{mfp}} f^{(1)} \sim K_n \frac{1}{\tau_{mfp}} f^{(0)}$$

$$\frac{\partial}{\partial t} f_0 \sim \frac{1}{\tau_S} f^{(0)} \sim K_n \frac{1}{\tau_{mfp}} f^{(0)}$$

$$\vec{v}\,\vec{\nabla}_r f_0 \sim \frac{1}{\tau_S} f^{(0)} \sim K_n \frac{1}{\tau_{mfp}} f^{(0)}$$

$$\vec{F}\,\vec{\nabla}_p f_0 \sim \frac{1}{\tau_S} f^{(0)} \sim K_n \frac{1}{\tau_{mfp}} f^{(0)}$$

Since on large time scales ($\sim \tau_S$) deviations from local equilibrium ($f^{(1)}$) occur in response to gradients and external forces, we can find a consistent truncation setting

$$f^{(1)}(t, \vec{r}, \vec{p}) \sim K_n\, f^{(0)}(t, \vec{r}, \vec{p})$$

$O(K_n^0)$: $\boxed{C[f^{(0)}]=0}$

$\Rightarrow f^{(0)}$ is local equilibrium distribution

$O(K_n^1)$: $\boxed{\delta C[f^{(0)},f^{(1)}] = \left(\frac{\partial}{\partial t} + \vec{v}\vec{\nabla}_r + \vec{F}\vec{\nabla}_p\right) f^{(0)}}$

$\Rightarrow$ solution for $f^{(1)}$ requires inversion of collision operator $\delta C[f^{(0)},f^{(1)}]$

Hilbert expansion can be developed to higher orders, which is e.g. required for the calculation of higher order transport coefficients

//

Discussion shows that solutions to BTE are hard to find even in the limit Kn≪1 where deviations from local equilibrium are small, as inversion of $SC[f^{(0)}, f^{(1)}]$ is highly non-trivial

In practice one often employs simplified versions of the collision Kernel, with the simplest physical one given by

Relaxation time approximation (RTA)

Expect that small deviations from equilibrium will relax exponentially towards equilibrium on a time scale

relaxation time: $\tau_R(T(t,\vec{r}), n(t,\vec{r}), \vec{p})$

suggests to approximate

$$\boxed{SC_{RTA}[f^{(0)}, f^{(1)}] = - \frac{f^{(1)}(t,\vec{r},\vec{p})}{\tau_R(T(\vec{r}), n(\vec{r}), \vec{p})}}$$

Even beyond the linearized description RTA is frequently employed as approximation of BTE. In this case one approximates the full collision integral as

$$C_{RTA}[f] = -\frac{f - f^{(0)}}{\tau_R}$$

where in this case the parameters $\vec{v}(t,\vec{r})$, $T(t,\vec{r})$, $n(t,\vec{r})$ characterizing the local equilibrium state $f^{(0)}$ have to be determined from the single particle distribution f, such that particle number, momentum and energy are conserved by $C_{RTA}[f]$

e.g. τ_R independent of $\vec{p}$

$$n[f](t,\vec{r}) = \int \frac{d^3p}{(2\pi\hbar)^3} f(t,\vec{r},\vec{p}) \longrightarrow n(t,\vec{r})$$

$$\vec{P}[f](t,\vec{r}) = \int \frac{d^3p}{(2\pi\hbar)^3} \vec{p} f(t,\vec{r},\vec{p}) \longrightarrow \vec{v}(t,\vec{r})$$

$$e_{kin}[f](t,\vec{r}) = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{\vec{p}^{\,2}}{2m} f(t,\vec{r},\vec{p}) \longrightarrow T(t,\vec{r})$$

//

Now that we have a systematic expansion for KinCL along with sufficiently simple approximations for $\delta C[f^{(0)}, f^{(1)}]$, will connect to macroscopic description of non-eq systems and discuss calculation of transport coefficients