Thermalization in Scalar Field Theories

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Based on
"Turbulent Thermalization" (R. Micha, I. Tkachev)
arXiv:hep-ph/0403101

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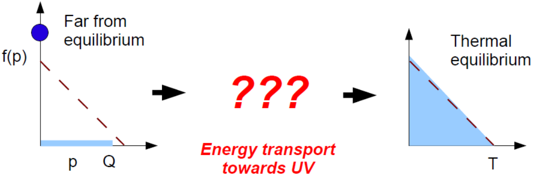
Connection to Inflation



[1]

- Equilibration of far-from-equilibrium systems finds many practical applications (heavy ion collisions or cosmology of the early Universe)
- At the end of inflation: energy stored in a Bose condensate
  $\rightarrow$ corresponding field: inflaton
- Highly unstable state: inflaton decays rapidly and explosively
- Inflaton decay stops when rate of interactions of created fluctuations (with each other and with the inflaton) is comparable to inflaton decay rate

[1] With the kind permission of Dr. Sören Schlichting

Describing Reheating

- Problems in describing reheating:
  - Very large initial occupation number
  - In many models: zero mode does not decay completely
- Therefore: simple perturbative approach is not valid
- But classical field theory is valid $\rightarrow$ study via classical lattice simulations

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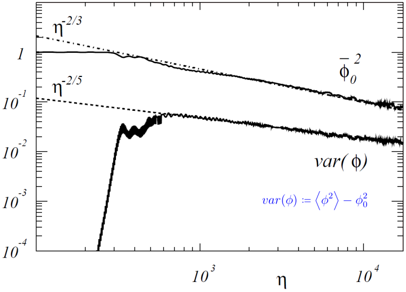
- Only one dynamical variable (Φ), whose initial homogeneous mode drives inflation
- End of inflation: when motion of homogeneous component changes from "slow-roll" to regime of oscillations
- Equation of motion after inflation in conformal coordinates ($\mathrm{d}s^2 = a(\eta)^2(\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2)$):

$$\Box\phi + \phi^3 = 0$$

- ϕ obtained from initial field by rescaling

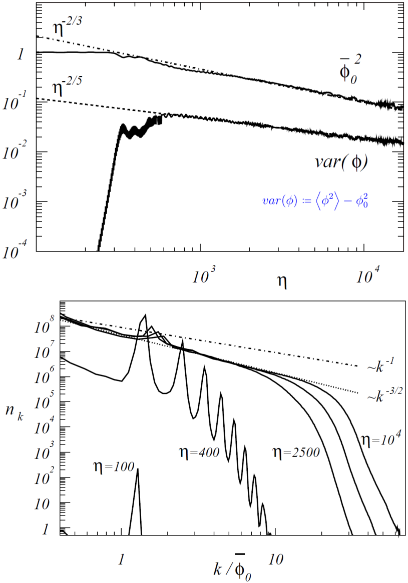
Massless $\lambda\Phi^4$-Model

Results for variance and amplitude and



Massless $\lambda\Phi^4$-Model

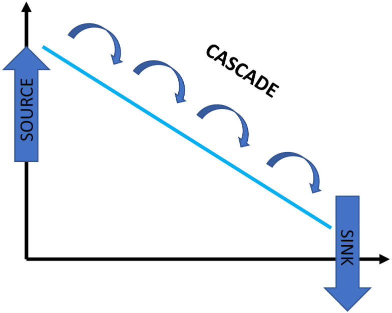
Results for variance and amplitude and occupation number



Massless $\lambda\Phi^4$-Model

Turbulence

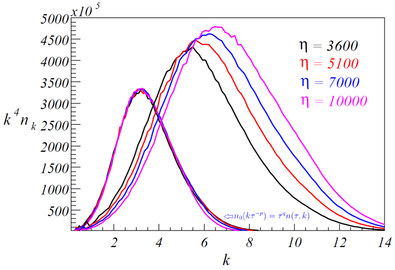
- Turbulence first discussed for fluids; appears also in systems of coupled waves (*wave turbulence*)
- Classification of turbulence:
  - *Driven (stationary) turbulence*: existence of active source of energy in momentum space
  - *Free (or decaying) turbulence*: freely propagating energy cascade after switch-off of active stage



Massless $\lambda\Phi^4$-Model

Regime of $\eta > 1500$

- Statistically close to a Gaussian distribution of field amplitudes and conjugated momenta
- In dynamically important region: $n_k \sim k^{-s}$ with $s = \frac{3}{2}$ ($\sim k^{-1}$ corresponds to thermal equilibrium)
- Cut-off at higher k; position moves towards the ultra-violet
- Motion describable as self-similar evolution: $n(k,\tau) = \tau^{-q} n_0(k\tau^{-p})$ with $\tau := \eta/\eta_c$
  - Best numerical fit: $q \approx 3.5p$, $p \approx 1/5$
  - p determines rate with which the system approaches equilibrium



Kinetic Theory or Lattice Simulation?

- At early times kinetic theory not applicable
  - Zero mode does not decay completely
  - Initially occupation numbers are of order $n_k \sim 1/\lambda$
- Lattice calculations limited in momenta and time range $\rightarrow$ apply kinetic theory at late time
- Compute universal scaling exponents within (weak) wave kinetic theory and compare with lattice computation
- At early times: dynamics driven by m-particle scattering with $m = 3$
- Wave kinetic turbulence gives in $d = 3$:

$$p = \frac{1}{2m-1} = \frac{1}{5} \quad , \quad s = d - \frac{m}{m-1} = \frac{3}{2}$$

Differences to More Complicated Models

- Expect turbulent stage and applicability of turbulence theory
- in $\lambda\phi^4$ model:
  - Preheating (parametric resonance) ends when half of inflaton energy is transferred
  - Inflaton energy decreases right after end of resonance stage
  - Followed by turbulent regime
- In models with $\#\text{fields} > 1$:
  - Turbulence should start at different time
  - Regime of free turbulence appears

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- Consider systems with spatially isotropic and homogeneous correlation functions (corresponds to cosmological conditions after inflation)
- Set-up: source of energy (or particles) at region k_{in}; sink at region k_{out}
- If source and sink are stationary: (eventual) development of stationary state with scale independent transport of conserved quantities
- Common features of reheating and turbulence:
  - Existence of localized source of energy at $k_{\text{in}} \sim k_{\text{res}}$ (oscillating inflaton zero-mode)
  - No other scale where energy is infused, accumulated or dissipated

- Differences between reheating and turbulence:
  - Non-existence of a sink
  - Source can be time-dependent
  - After complete inflaton decay: neither source nor sink exists
- However:
  - First point: driven turbulent flux of energy will be established in some "inertial" range $k_{\text{in}} < k < k_{\text{out}}$; flux of energy is constant throughout inertial range, i.e. $E(t) \sim t$
  - Second point: time dependent source changes picture dramatically; weak time dependence can be handled and allows "close-to-stationary" and "close-to-turbulent" evolution
  - Third point: particle distribution in inertial range still close to turbulent power laws; collision integral should approach a minimum; results in same shape for particle distribution

- Dynamics of coupled waves close to a stationary state described by wave kinetic equation:
$$\dot{n}_k = I_k[n] \quad \text{where} \quad I_k[n] = \int \mathrm{d}\Omega(k, q_i) F(k, q_i)$$
- In classical limit: $F(\zeta n) = \zeta^{m-1} F(n)$ (for interaction of m particles)
- Consider here only energy and particle density as conserved quantities (there could be more)
- Stationary turbulence: energy flux, $S^\rho(r) \sim \int^r \mathrm{d}k k^{d-1} \omega_k I_k[n]$, should be scale invariant, i.e. independent of integration limit r
$\rightarrow$ find conditions s.t. $S^\rho(r) = S^\rho$

- Consider states with $I_{\xi k}[n] = \xi^{-\nu} I_k[n]$, i.e. $I_k[n] = k^{-\nu} I_1[n]$
- Consider dispersion law is homogeneous function: $\omega(\xi k) = \xi^{\alpha} \omega(k)$
- All in all: $S^{\rho}(r) \sim -r^{d+\alpha-\nu} \frac{I_1(\nu)}{d+\alpha-\nu}$
  $\Rightarrow$ flux is scale invariant for $\nu = d + \alpha$
- From now on: consider $\mathrm{d}\Omega$ is homogeneous function with exponent μ
- consider $n(q) \sim q^{-s}$, i.e. $F(\xi k, \xi q_i) = \xi^{-s(m-1)} F(k, q_i)$, i.e. $I_{\xi k} = \xi^{\mu - s(m-1)} I_k$
- This gives scaling of particle distribution in turbulent states with constant energy transport (i.e. in energy cascade):
$$s = \frac{d + \alpha + \mu}{m - 1}$$

- Assume self-similar evolution for describing e.g. free turbulent
- Describe as rescaling of momenta accompanied by suitable change of the overall normalization: $n(k,\tau) = A^\gamma n_0(kA)$
- Wave kinetic equation gives:

$$A = \Theta^{-p} \quad , \quad \Theta \coloneqq \frac{\Gamma t_0}{p} \int_1^\tau \mathrm{d}\tau' B(\tau') + 1 \quad , \quad p \coloneqq \frac{1}{\gamma(m-2) - \mu}$$

- p determines the speed of motion over momentum space of the distribution function $\rightarrow$ defines e.g. time scale of thermalization

- Time-independent background: $B = 1$
- Specify γ by boundary conditions
  - Isolated systems:

$$\text{rel.:}\ \ p_i = \frac{1}{(d+\alpha)(m-2)-\mu} \quad , \quad \text{non-rel.:}\ \ p_i = \frac{1}{d(m-2)-\mu}$$

  - Driven turbulence:

$$p_t = (m-1)p_i$$

  - Non-stationary source:

$$p = (1 + r(m-2))p_i \quad \text{with} \quad E(\tau) = E_0 \tau^r$$

- Time-dependent background: $B(\tau) = \tau^{-\kappa}$
- Solution in this case: $\Theta = \Theta(\tau^{1-\kappa})$
- Late time behaviour depends on sign of $1-\kappa$
  - $1-\kappa > 0$: distribution propagates to ultraviolet without bound, i.e. $A \sim \tau^{(1-\kappa)p}$
  - $1-\kappa < 0$: $A(\tau)$ approaches finite limit $A(\tau = \infty) = \left[1 + \frac{1}{\kappa - 1}\right]^{-p}$, i.e. propagation of particle distribution towards ultraviolet is limited

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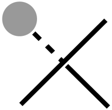
- Example: $\lambda\phi^4$-model
- Scaling exponents for energy cascade in isolated systems within this model:

$$\text{rel.:} \quad p_i = \frac{1}{(2m-1)} \quad , \quad \text{non-rel.:} \quad p_i = \frac{1}{2}$$
$$\text{rel.:} \quad s = d - \frac{m}{m-1} \quad , \quad \text{non-rel.:} \quad s = d$$

- 3-particle scattering in $\lambda\phi^4$ (appears when interaction with zero-mode is important) gives in $d = 3$: $p_i = \frac{1}{5}$, $s = \frac{3}{2}$
  $\rightarrow$ coincides with numerical values



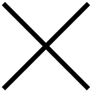
[2]

[2] With the kind permission of Dr. Sören Schlichting

- Example: $\lambda\phi^4$-model in $d = 3$ if zero-mode is absent (late times)
- Scaling exponents for energy cascade in isolated systems within this model:

$$p_i = \frac{1}{(d+1)(m-2)-1} \quad , \quad s = \frac{d+2}{m-1}$$

- In $d = 3$ for 4-particle processes: $p_i = \frac{1}{7}$, $s = \frac{5}{3}$



[3]

[3] With the kind permission of Dr. Sören Schlichting

- Self-similar evolution modified if explicit time dependence is present
- Relativistic regime:
  - Time-dependence enters vial coupling to zero-mode
  - Leads to new specific terms in collision integral
  - $s = \frac{3}{2}$ still applicable as collision integral is dominated by 3-particle interaction
  - p changes because amplitude of zero-mode changes with time; during initial stage: driven turbulence, i.e. $p_t = 2p_i$; at late times: $p \approx \frac{1}{5}$ during integration time with deviation of 5%

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Two Interacting Fields

The Model

- At end of inflation the Universe is very close to spatially flat Friedmann model
- Consider massless fields
- Conformal transformation allows mapping of dynamics in expanding Friedmann Universe into case of Minkowski space-time
- Dynamics obtained from Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial\Phi)^2 + \frac{1}{2}(\partial X)^2 - V(\Phi, X),$$

$$V(\Phi, X) = \frac{\lambda_\Phi}{4}\Phi^4 + \frac{\lambda_{\Phi X}}{2}\Phi^2 X^2 + \frac{\lambda_X}{4}X^4$$

- Φ identified with the inflaton
- Equation of motion in dimensionless form after rescaling:

$$\begin{cases} \Box\phi + \phi^3 + g\chi^2\phi = 0 \\ \Box\chi + h\chi^3 + g\phi^2\chi = 0 \end{cases}$$

$$g := \frac{\lambda_{\Phi X}}{\lambda_\Phi} \quad , \quad h := \frac{\lambda_X}{\lambda_\Phi}$$

Two Interacting Fields

Numerical Results - Particle Spectra

- Set $g = 30$; vary h; duration and relative importance of different regimes (parametric resonance, driven turbulence, free turbulence) influenced by different values of h
- At late times: very similar behaviour to case with one field
- Same scaling exponents ($s = \frac{3}{2}$, at sufficiently late times: $p = \frac{1}{5}$)

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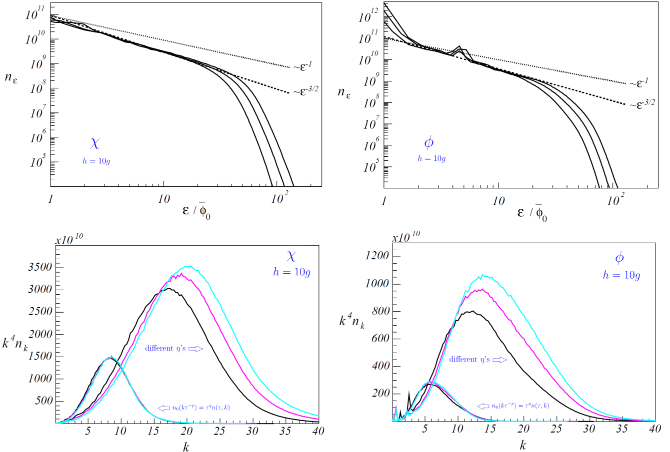
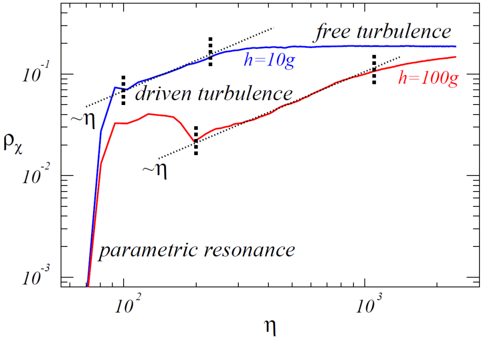


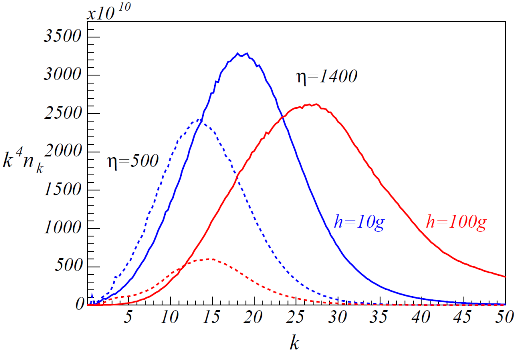
Numerical Results - Driven and Free Turbulence in Two Field Model

- Driven turbulence expected to appear when $g \gg 1$, $h \gg 1$, i.e. when parametric resonance stops early
- Use energy densities instead of particle distributions to describe dynamics



Numerical Results - Driven and Free Turbulence in Two Field Model

- Thermalization proceeds faster with larger couplings



- In systems with acceptable reheating temperature, parametric resonance stops only when negligible fraction of inflaton energy has decayed
  $\Rightarrow$ Driven turbulence is major mechanism of energy transfer from inflaton into particles

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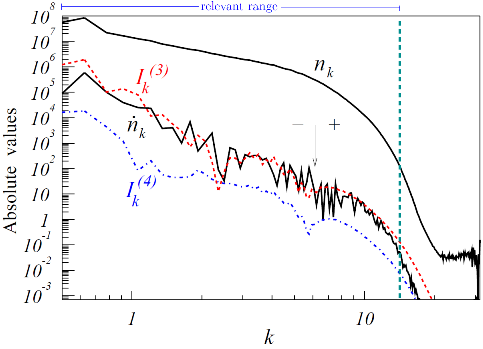
Applicability of Kinetic Approach

3- or 4-Particle Interaction?

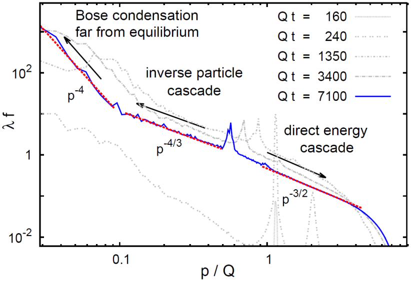
- Observed exponents $s \approx \frac{3}{2}$, $p = \frac{1}{5}$ are corresponding to 3-particle interactions
- Problem: no bare 3-particle interactions in the considered systems
  - Appear effectively in interaction with zero-mode; collision integral multiplied by amplitude of zero-mode squared
  - Amplitude decays, i.e $p = \frac{1}{5}$ only valid in small interval
- 4-particle interaction responsible for observed scaling?
  - $p_i = \frac{1}{7}$ for 4-particle interaction (not that far away)
  - Expected value $s = \frac{5}{3}$ far away
  - 4-particle scattering dominates when variance of fluctuations larger than ϕ_0^2; not the case here
- Is weak wave turbulence theory applicable?
  - Study collision integrals (and correlators)

- Strategy: compute collision integrals numerically with n_k obtained from lattice calculations; calculate time derivatives of distribution function using lattice data and check relation $\dot{n}_k = I_k[n]$



Applicability of Kinetic Approach

Problems And Open Questions



[4]

- Slow decay of inflaton at late times
- Numerical calculation on larger lattice shows strong turbulence in infra-red
  $\Rightarrow$ standard perturbation theory not applicable

[4] Berges, J. et al., Journal of High Energy Physics 2014.5 (2014), arXiv:1312.5216

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- During preheating many effects occur, with common origin: rapid particle creation and large fluctuations
  - These are unaffected by the obtained results
- Sometimes it is necessary to trace events further in time (e.g. to find out when thermal equilibrium will be established)
- Main interest: field variances and problem of thermalization

- Field variances may give answer to symmetry restoration
- For negligible anomalous correlators:

$$var(\chi,\tau) \coloneqq \langle \chi^2 \rangle - \langle \chi \rangle^2 = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{n_k}{\omega_k} = \tau^{-\gamma p_i} var_0(\chi)$$

- Relativistic regime:
  - Regime of driven turbulence: $var(\chi) = \tau^{p_i} var_0(\chi) = \tau^{\frac{1}{5}} var_0(\chi)$
  - Regime of free turbulence: $var(\chi) = \tau^{-2p_i} var_0(\chi) = \tau^{-\frac{2}{5}} var_0(\chi)$ (for late times $p_i = \frac{1}{7}$ because 4-particle scattering dominates)
- Non-relativistic regime:
  - $var(\chi,\tau) =$ const. in case of both driven and free turbulence
- In regime of driven turbulence: variance varies slowly; energy in particles grows fast
  - In agreement with theory: variance can be large after parametric resonance stage; amount of energy transferred during this stage is low
  - Energy transfer occurs in regime of driven turbulence

- Consider self-similar evolution; introduce specific models via factor $A(\tau)$
- Classical evolution stops when system approaches quantum regime (when occupation number becomes of order one)
- Consider here only free turbulence
- Estimate of thermalization time: $\tau^{\text{th}} \sim (k_f/k_i)^{1/p}$
- Consider evolution of sub-system of excitations of a field χ
    - Initially fixed part of energy deposited into it; since then χ evolves as isolated system

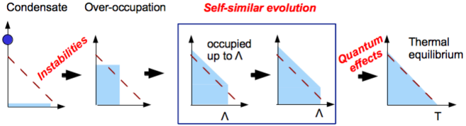
Physical Application

Thermalization in Absence of Zero-Mode

- Treat expansion of Universe in conformal coordinates
- Let c_χ: fraction of inflaton energy deposited into field χ during preheating and driven turbulence ($\rho_f = \rho_i = c_\chi \rho_{\text{tot}}$)
- Relativistic regime:
  - Thermalization time:

$$\tau^{\text{th}} \sim c_\chi^{7/4} \left[\frac{M_{\text{Pl}}}{M_\phi} \right] \sim c_\chi^{7/4} \lambda^{-7/4} \text{ with } M_\phi = \sqrt{\lambda} M_{\text{Pl}}$$

- Relativistic and non-relativistic results coincide with theoretical estimates from Minkowski space-time and Friedmann Universe



[5]

[5] With the kind permission of Dr. Sören Schlichting

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Conclusion and Open Questions

- Process of thermalization of classical systems in non-equilibrium state can be divided into three stages
- Initial stage of preheating powered by *parametric resonance*
  - Energy in particles grows exponentially
  - Amount of energy transferred depends on coupling strength
- Parametric resonance followed by *driven turbulence*
  - Energy in particles grows linearly
  - Stops when energy in particles starts to dominate the overall energy balance
  - Major mechanism of energy transfer from inflaton zero-mode into particles
- Last stage classified as *free turbulence*
  - Energy in particles conserved during this stage
  - Particle distribution obeys self-similar evolution
  - Continues until quantum regime is reached
- Open Questions:
  - Late time decay of inflaton
  - Thermalization in quantum regime

Thank You!

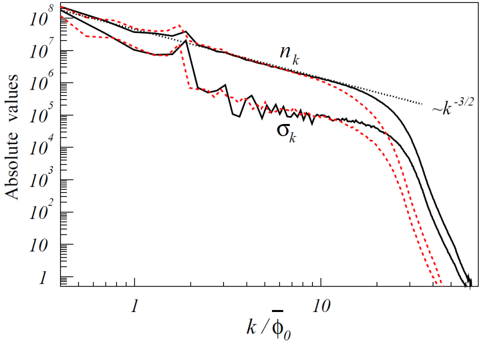
Further Reading

- http://kaiden.de/data/ICTS19/SS_ICTS_L3.pdf
- S. Nazarenko; Wave turbulence
- Berges, J. et al. "Basin of Attraction for Turbulent Thermalization and the Range of Validity of Classical-Statistical Simulations.", Journal of High Energy Physics 2014.5 (2014)

- Non-relativistic regime:
  - Time-dependence enters via expanding Universe
  - Time-dependence given by $B(\tau) = [bH_0\eta_0(\tau - 1) + 1]^{-\kappa}$
  - Scaling exponent: $p = \frac{1}{3(m-2)-\mu}$
  - For $\tau \to \infty$: $k_c(\tau = \infty) = \frac{1}{[b(\kappa-1)H_0\eta_0]^p} k_c(1)$, i.e. thermal equilibrium can only be reached for $H_0\eta_0 < 1$
  - However, $k_c(\tau = \infty)$ must not be smaller than typical momentum in equilibrium to reach thermal equilibrium

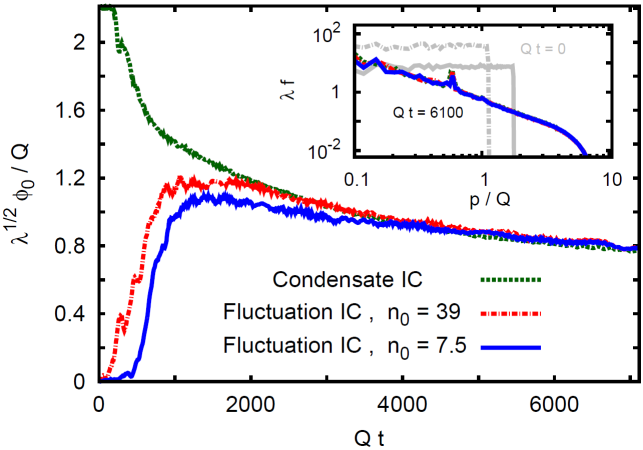
Anomalous Correlators σ_k

- Usual assumption for deriving kinetic equations: $\langle a_k a_q \rangle \ll \langle a_k^* a_q \rangle$
- Problem: assumption does not always hold
  - If coherent processes are important, correlators modify dynamics of n_k and should be included in kinetic equations
- Therefore: check condition $|\sigma_k| \ll n_k$ since σ_k were neglected



- Non-relativistic regime:
  - Thermalization time: $\tau^{\text{th}} \sim \left[c_\chi \frac{M_\phi}{M_X}\right]^{2/3} \lambda^{-2/3}$
  - X: Number of particles

Dependence on Initial Conditions



[6]

[6] Berges, J. et al., Journal of High Energy Physics 2014.5 (2014), arXiv:1312.5216